Theory and Application of the Bivariate Credibility Model

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In recent times, the insurance activity had shown an implausible increase which had an expressively valuable effect on the economic growth of several countries globally. Services that influence growth in the country include the deployment of a colossal sum of funds by means of premiums for short- and long-term investment for development and underwriting of risk in economic entities. We propose in this paper new credibility premiums, which is based on a relationship between the number and the number of claims of a contract for that year, under the irreducible random variables, which helps us ensure the covariance matrix inversion. And then we calculate bivariate Bühlmann and Bühlmann Straub estimators with application. Thus, we arrive at the new estimators of the individual premium by using additional sources of data. We conclude with structure parameters estimators and application.
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1 Introduction

Credibility theory is an actuarial pricing method. Due to this theory the insurer may calculate the premium of a portfolio of contracts. Credibility theory is to combine the available data on the portfolio and each individual (the contract) see [11], [12]. Several models of credibility have been established [1], [2], [3]. For that we propose in this paper to present new premiums credibility of the insurance contract. We extend the univariate credibility model to the bivariate credibility theory by using additional sources of data. In many cases we find that insurance there is a relationship between the number and the number of claims of a contract for a single period see [5]. We write new estimators of individual premium.

The indispensable problem of linear credibility is determined estimator of the individual premium for a contract or a portfolio of insurance contracts [7],[8]. As the credibility estimator is linear in the observations of the portfolio. To determine a precise form of the estimator of credibility, we must solve the problem of minimizing the least squares sense:

$$\Psi = min_{\{\zeta^1,\zeta^2,\xi^1,\xi^2\}} E\left[\begin{pmatrix}\mu^1(\Theta) - \hat{\mu}^1(\zeta^1,\zeta^2) \\ \mu^2(\Theta) - \hat{\mu}^2(\xi^1,\xi^2)\end{pmatrix}^2\right] \qquad (1)$$

With $\mu^1(\Theta)$ and $\mu^2(\Theta)$ are respectively, the conditional expectation of the number of claims, and the conditional expectation of number of claims, and Θ is a risk profil. Here, $\hat{\mu}^1(\zeta^1,\zeta^2)$ and $\hat{\mu}^2(\xi^1,\xi^2)$ are linear forms which will be specified in the various models, andζ^1,ζ^2, ξ^1,ξ^2 are credibility factors. The goal in greatest accuracy credibility consists in finding the closest (in the mean square sense) estimator of the aggregate risk premium.

This paper is structured in five parts.

2 Preliminaries

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $L^2 = L^2(\Omega,\mathcal{F},\mathbb{P})$ the Hilbert space of square-integrable random variables.

The task of credibility theory is to provide decisions or estimators for future uncertain events. Deterministic or certain events do not need to be estimated; they simply can be calculated. It is therefore reasonable to eliminate deterministic dependencies within the random structures regarded. On the other hand, credibility theory is restricted to find linear estimators using the Hilbert space structure of L^2, in particular its orthogonality. Thus, it is sufficient to eliminate linear deterministic dependencies in the setup of credibility. This leads to the following definition:

Definition 2.1
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^2 = L^2(\Omega, \mathcal{F}, \mathbb{P})$ the Hilbert space of square integrable random variables. A sequence of random variables $X = (X_1, \ldots, X_n)$ is called (linearly) reducible with respect to the probability measure $\mathbb{Q}$ if there exist real coefficients α_0 and $\alpha = (\alpha_1, \ldots, \alpha_n) \neq 0$ such that :

$$\mathbb{Q}(\sum_{i=1}^{n} \alpha_i X_i = \alpha_0) = 1. \quad (2)$$

In the context of $\mathbb{Q}$ -square-integrable random variables the irreducibility of a sequence of random variables can be characterized as follows:

Proposition 2.1
A sequence $X = (X_1, \ldots, X_n)$of $\mathbb{Q}$ -square-integrable random variables are irreducible if and only if their $\mathbb{Q}$ -covariance matrix

$$Cov_{\mathbb{Q}}(X) := \left((X_i, X_j)\right)_{1 \le i,j \le n} \quad (3)$$

is positive definite.

Proof.
Let $X = (X_1, \ldots, X_n) \in (L^2(\mathbb{Q}))^n$.
The matrix $Cov_{\mathbb{Q}}(X)$ not being positive definite is equivalent to having 0 as spectral value. The later is equivalent to:
$\alpha^{tr} Cov_{\mathbb{Q}}(X)\alpha = Var_{\mathbb{Q}}(\sum_i \alpha_i X_i) = 0$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \neq 0$.
But the vanishing of the $\mathbb{Q}$ -variance is again equivalent to $E_{\mathbb{Q}} \sum_i \alpha_i X_i = \alpha_0$, i.e. the reducibility of X.

3 The univariate credibility model

In this section, we present the standard Bühlmann model (1967) see [4],[8] for one no life insurance contract. We denote X_j (j =1,...,n) are a random variables present the observations of a contract, and Θ a risque profil.

Assumptions (univariate credibility model)

i)The random variables X_j (j =1,...,n) are, conditional on $\Theta = \theta$, independent with the same distribution function F_θ with the conditional moments

$$\mu(\theta) = E\left[X_j \,|\Theta = \theta\right], \quad (4)$$

$$\sigma^2(\theta) = Var[X_j \,|\Theta = \theta] \quad (5) .$$

ii) Θ is a random variable with distribution $U(\theta)$. In this model the objective is to find an estimator for the individual premium $\mu(\theta)$, which are linear in the observations. So, the best linear estimator has to be of the form:

$$\widehat{\mu(\theta)} = \hat{a} + \hat{b}\bar{X} \quad (6)$$

Where $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ and $\hat{a}, \hat{b}$ are the solution of the minimizing problem:

$$E\left[\left(\mu(\theta) - \hat{a} - \hat{b}\bar{X}\right)^2\right] = min_{a,b\epsilon\mathbb{R}}\, E\left[\left(\mu(\theta) - \hat{a} - \hat{b}\bar{X}\right)^2\right]$$

After the partial derivatives with respect to a, resp. b, and the dependency structure imposed by Model Assumptions, we get the following theorem:

Theorem3.1
The univariate credibility estimator under model Assumptions is given by

$$\widehat{\mu(\Theta)} = (1-\alpha)\mu_0 + \alpha\bar{X} \quad (7)$$

where

$$\mu_0 = E[\mu(\Theta),$$

$$\alpha = \frac{n}{n + \sigma^2/_{\upsilon^2}}.$$

$$\upsilon^2 = Var[\mu(\Theta)] \quad (8)$$

and
$\sigma^2 = E[\sigma^2(\Theta)]$. (9)
See([4],[7])
In the next section, we present a generalization of univariate credibility model, we bascule to the bivariate Bühlmann model.

4 The bivariate Bühlmann model

We consider an insurance portfolio consisting of n ≥ 2 contracts. Each contract i has been activated in the last t periods at least for one period.
By T(i) ⊆{1,...,t} we denote the set of the activated past periods of contract i, assuming that its cardinality t_i := |T(i)|≥ 2.We set $t_0 := \sum_{1\le i\le n} t_i$. In order to be able to include the future period t +1, we set $T^+(i)$:= T(i) ∪{t +1}. Now we have for each contract i the set of pairs of random variables:

$$\left(X_i{}^1, X_i{}^2\right) = \left(X_{i,s}{}^1, X_{i,s}{}^2\right)_{s\in T^+(\mathrm{i})},$$

i =1,...,n. For example, $X_{i,s}{}^1$ *and* $X_{i,s}{}^2$ can be interpreted as respectively the amount and the number of claims of the contract i in the period s.
The random variables of the whole model are denoted by :
$(X^1, X^2) = \left(X_i{}^1, X_i{}^2\right)_{i=1,\dots,n}$, (10)
We make the following model assumptions:
(B1) The random structure variables $(\Theta_i)_{i=1,\dots,n}$ have the identical distribution $\mathbb{P}_\Theta$.
(B2) The families of random variables $(\Theta_i, X_i{}^1, X_i{}^2)_{i=1,\dots,n}$, are mutually non-correlated with
$X_{i,s}{}^1, X_{i,s}{}^2 \in L^2$ for all s ∈ $T^+(i)$, i =1,...,n. The complete set of random variables
$\{X_{i,s}{}^1, X_{i,s}{}^2$ i,s | s ∈ $T^+(i)$,i =1,...,n} is assumed to be ℙ -irreducible.
(B3) For fxed i and conditioned by Θ_i, the pairs of random variables $\{X_{i,s}{}^1, X_{i,s}{}^2$ i,s | s ∈ $T^+(i)$,i =1,...,n} are mutually non-correlated with the first and second conditional moments depending only on Θ_i.
τ =1, 2. Moreover, on a non-negligible set the pair $(X_{i,s}{}^1, X_{i,s}{}^2)$is assumed to be $\mathbb{P}_{(\cdot|\Theta i)}$-irreducible:

$$\mathbb{P}_\Theta\{\omega | X_{i,s}{}^1, X_{i,s}{}^2 \text{ is } \mathbb{P}(.|\Theta_i(\omega)) - \text{irreducible}\} > 0.$$

For conditional and non-conditional moments,we save same notations as in [5].
A first result from the assumptions made is the following:
Lemma 4.1.
Under the model assumptions **(B1)-(B3)** we have:

$$D_{i,1} := (t_i B^{(1)} + A^{(1)})(t_i B^{(2)} + A^{(2)}) - (t_i L + K)^2 > 0 \qquad (11)$$

and

$$D_2 := A^{(1)} A^{(2)} - K^2 > 0. \quad (12)$$

Proof
To show (11), we know by assumption **(B1)** that the sequence of random variables $(X_i{}^1, X_i{}^2)$is assumed to be $\mathbb{P}_{(\cdot|\Theta i)}$ irreducible, and so is the pair vector $(\sum_{s\in T(i)} X_{i,s}{}^1, \sum_{s\in T(i)} X_{i,s}{}^2)$. Lemma2.2 tells us that their covariance matrix is positive definite:

$$Cov\left(\sum_{s\in T(i)} X_{i,s}{}^1, \sum_{s\in T(i)} X_{i,s}{}^2\right) = \begin{pmatrix} t_i B^{(1)} + A^{(1)} & t_i L + K \\ t_i L + K & t_i B^{(2)} + A^{(2)} \end{pmatrix}$$

This implies that the determinant
$D_{i,1} := (t_i B^{(1)} + A^{(1)})(t_i B^{(2)} + A^{(2)}) - (t_i L + K)^2$
Of the last matrix is strictly positive which the statement is in (11).
For the inequality (12), the assumption **(B3)** implies with Lemma2.2 that on a set of positive $\mathbb{P}_\Theta$-measure the covariance matrix with respect to the conditional probability $\mathbb{P}_{(\cdot|\Theta i)}$ of the pair $(X_{i,s}{}^1, X_{i,s}{}^2)$ is positive

definite:

$$Cov_{\mathbb{P}(\cdot|\Theta i)}\left(X_{i,s}{}^{1}, X_{i,s}{}^{2}\right) = \begin{pmatrix} \alpha^{(1)}(\Theta) & \kappa(\Theta) \\ \kappa(\Theta) & \alpha^{(2)}(\Theta) \end{pmatrix}.$$

This implies that for any pair(α_1,α_2) $\neq$0 the conditional variance $Var_{\mathbb{P}(\cdot|\Theta i)}$ ($\alpha_1 X_{i,s}{}^{1}+$ $\alpha_2 X_{i,s}{}^{2}$)$\geq$ 0 is strictly positive on a set of positive $\mathbb{P}_{\Theta}$-measure. Therefore also

$$\mathrm{E}_{\Theta}\left(\mathrm{Var}_{\mathrm{P}(\cdot|\Theta_i)}\left(\alpha_1 X_{i,s}{}^{1} + \alpha_2 X_{i,s}{}^{2}\right)\right) = (\alpha_1 \quad \alpha_2)\begin{pmatrix} \mathrm{A}^1 & \mathrm{K} \\ \mathrm{K} & \mathrm{A}^2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} > 0.$$

This shows that the matrix $\begin{pmatrix} \mathrm{A}^1 & \mathrm{K} \\ \mathrm{K} & \mathrm{A}^2 \end{pmatrix}$ is positive definite and consequently its determinant $D_2 := A^{(1)}A^{(2)} - K^2$ is strictly positive.

We want to find a linear credibility estimator of the bivariate system $(X_{i,t+1}{}^{1}, X_{i,t+1}{}^{2})$. Consequently, it is assumed that such estimators of $X_{i,t+1}{}^{(\tau)}$, τ =1, 2, have the form

$$\hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^1, \mathrm{X}^2) = \zeta_{\mathrm{i},0}{}^{(\tau)} + \sum_{1<j<n}\sum_{s\in \mathrm{T(j)}}\left(\zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)} \quad X_{j,s}{}^{(\tau)} + \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)} \quad X_{j,s}{}^{(3-\tau)}\right) \qquad (16)$$

With real coefficients $\zeta_{\mathrm{i}}{}^{(\tau)} = \left(\zeta_{\mathrm{i},0}{}^{(\tau)}, \zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)}, \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)}\right)$.

Theorem4.1.
Under the assumptions **(B1)–(B3)** the optimal solution to the linear credibility problem is:

$$\hat{\mu}^{(\tau)*}{}_{i}\,(\mathrm{X}^1{}_{\mathrm{i}}, \mathrm{X}^2{}_{\mathrm{i}}) = \left(1 - \zeta_{\mathrm{i}}{}^{(\tau,\tau)}\right)M^{(\tau)} + \frac{1}{\mathrm{t_i}}\sum_{s\in \mathrm{T(i)}}\left(\zeta_{\mathrm{i}}{}^{(\tau,\tau)} \quad X_{i,s}{}^{(\tau)} + \zeta_{\mathrm{i}}{}^{(\tau,3-\tau)}\left(\quad X_{i,s}{}^{(3-\tau)} - M^{(3-\tau)}\right)\right) \quad (13)$$

τ =1, 2, where the credibility coefficients $\zeta_{\mathrm{i}}{}^{(\tau,\tau)}, \zeta_{\mathrm{i}}{}^{(\tau,3-\tau)}$ are given by the following formulas:

$$\zeta_{\mathrm{i}}{}^{(\tau,\tau)} = \frac{\mathrm{t_i}}{\mathrm{D_{i,1}}}\left(\mathrm{B}^{(\tau)}\left(\mathrm{t_i}\mathrm{B}^{(3-\tau)} + \mathrm{A}^{(3-\tau)}\right) - \mathrm{L}(\mathrm{t_i}\mathrm{L} + \mathrm{K})\right) \qquad (14)$$

$$\zeta_{\mathrm{i}}{}^{(\tau,3-\tau)} = \frac{\mathrm{t_i}}{\mathrm{D_{i,1}}}\left(\mathrm{A}^{(\tau)}\mathrm{L} - \mathrm{B}^{(\tau)}\mathrm{K}\right). \qquad (15)$$

Remark 4.1
In contrast to the standard credibility formula, here the credibility factors $\zeta_{\mathrm{i}}{}^{(\tau,\tau)}, \zeta_{\mathrm{i}}{}^{(\tau,3-\tau)}$ are not necessarily in [0, 1].

Proof. :
Obviously, the credibility problem:

$$min_{\zeta_{\mathrm{i}}{}^{(1)},\zeta_{\mathrm{i}}{}^{(2)}} E\left[\sum_{\tau=1,2}\left(X_{i,t+1}{}^{(\tau)} - \hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^1, \mathrm{X}^2)\right)^2\right] = \sum_{\tau=1,2} min_{\zeta_{\mathrm{i}}{}^{(\tau)}}\ E[\left(X_{i,t+1}{}^{(\tau)} - \hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^1, \mathrm{X}^2)\right)^2].\,(16)$$

Can be separated. As necessary conditions of optimality, we have for i,j ≤ n and s ∈ T(j):.

$$0 = \frac{\partial}{\partial \zeta_{\mathrm{i},0}{}^{(\tau)}} = E\left[X_{i,t+1}{}^{(\tau)} - \zeta_{\mathrm{i},0}{}^{(\tau)} - \sum_{1<j<n}\sum_{s\in \mathrm{T(j)}}(\zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)} \quad X_{j,s}{}^{(\tau)} + \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)} \quad X_{j,s}{}^{(3-\tau)})\right].$$

$$0 = \frac{\partial}{\partial \zeta_{i,j,s}{}^{(\tau,\tau)}}\Psi = E\left[X_{j,s}{}^{(\tau)}(X_{i,t+1}{}^{(\tau)} - \zeta_{i,0}{}^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in T(j')}(\zeta_{i,j',s'}{}^{(\tau,\tau)}\ X_{j',s'}{}^{(\tau)} + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\ X_{j',s'}{}^{(3-\tau)}))\right].$$

$$0 = \frac{\partial}{\partial \zeta_{i,j,s}{}^{(\tau,3-\tau)}}\Psi = E\left[X_{j,s}{}^{(3-\tau)}(X_{i,t+1}{}^{(\tau)} - \zeta_{i,0}{}^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in T(j')}(\zeta_{i,j',s'}{}^{(\tau,\tau)}\ X_{j',s'}{}^{(\tau)} + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\ X_{j',s'}{}^{(3-\tau)}))\right].$$

It follows that $\zeta_{i,0}{}^{(\tau)} = M^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in T(j')}(\zeta_{i,j',s'}{}^{(\tau,\tau)}M^{(\tau)} + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}M^{(3-\tau)})$. Replacing $\zeta_{i,0}{}^{(\tau)}$ in (22) it turns out that we can pass to centralized moments.

The last two lines in (22) are now equal to:

$$0 = \delta_{i,j}B^{(\tau)} - \sum_{1<j'\leq n}\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(B^{(\tau)} + \delta_{s,s'}A^{(\tau)}\right) + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(L + \delta_{s,s'}K\right)\right)$$

$$0 = \delta_{i,j}B^{(\tau)} - \sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(B^{(\tau)} + \delta_{s,s'}A^{(\tau)}\right) + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(L + \delta_{s,s'}K\right)\right) \quad (17)$$

$$0 = \delta_{i,j}L - \sum_{1<j'\leq n}\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(L + \delta_{s,s'}K\right) + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(B^{(3-\tau)} + \delta_{s,s'}A^{(3-\tau)}\right)\right).$$

$$0 = \delta_{i,j}L - \sum_{s'\in T(j')}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(L + \delta_{s,s'}K\right) + \zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(B^{(3-\tau)} + \delta_{s,s'}A^{(3-\tau)}\right)\right)$$

Setting $\zeta_{i,j'}{}^{(\tau,3-\tau)} = \sum_{s'\in T(j')}\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\tau = 1, 2$, the summation over s ∈ T(j) yields

$$\delta_{i,j}\ t_jB^{(\tau)} = \zeta_{i,j}{}^{(\tau,\tau)}\left(t_jB^{(\tau)} + A^{(\tau)}\right) + \zeta_{i,j}{}^{(\tau,3-\tau)}\left(t_jL + K\right)$$
$$\delta_{i,j}\ t_jL = \zeta_{i,j}{}^{(\tau,\tau)}\left(t_jL + K\right) + \zeta_{i,j}{}^{(\tau,3-\tau)}\left(t_jB^{(3-\tau)} + A^{(3-\tau)}\right)$$

By (14) the determinant of the linear system (28) is positive such that $0 = \zeta_{i,j}{}^{(\tau,\tau)} =$

$\zeta_{i,j}{}^{(\tau,3-\tau)}$ for i ≠ j and $\zeta_i{}^{(\tau,\tau)} := \zeta_{i,i}{}^{(\tau,\tau)}$ and $\zeta_i{}^{(\tau,3-\tau)} := \zeta_{i,i}{}^{(\tau,3-\tau)}$ as given in (19). Knowing $\zeta_{i,j}{}^{(\tau,\tau)}$ and $\zeta_{i,j}{}^{(\tau,3-\tau)}$, (26) gives in addition

$$\left(\delta_{i,j} - \zeta_{i,j}{}^{(\tau,\tau)}\right) B^{(\tau)} - \zeta_{i,j}{}^{(\tau,3-\tau)} L = \zeta_{i,j,s}{}^{(\tau,\tau)} A^{(\tau)} + \zeta_{i,j,s}{}^{(\tau,3-\tau)} K$$
$$\left(\delta_{i,j} - \zeta_{i,j}{}^{(\tau,\tau)}\right) L - \zeta_{i,j}{}^{(\tau,3-\tau)} B^{(3-\tau)} = \zeta_{i,j,s}{}^{(\tau,\tau)} K + \zeta_{i,j,s}{}^{(\tau,3-\tau)} A^{(3-\tau)}$$

And since the determinant D_2 of this system is positive by (15), the unique solution must be $\zeta_{i,j,s}{}^{(\tau,\tau)} = \frac{\zeta_{i,j}{}^{(\tau,\tau)}}{t_j}$ and $\zeta_{i,j,s}{}^{(\tau,3-\tau)} = \frac{\zeta_{i,j}{}^{(\tau,3-\tau)}}{t_j}$. ■

Since in applications the parameters appearing in Theorem 3.2 are unknown, the following standard moment estimators are generally used.

Theorem4.2
(i). For the parameters M , A , K, and L, (τ =1, 2), the following respective estimators are unbiased:

$\widehat{M}^{(\tau)} := \frac{1}{t_0} \sum_{1<i\le n} \sum_{s\in T(i)} X_{i,s}{}^{(\tau)}$,

$$\hat{A}^{(\tau)} := \frac{1}{t_0 - n} \sum_{1<i\le n} \sum_{s\in T(i)} \left(X_{i,s}{}^{(\tau)} - \overline{X_\iota{}^{(\tau)}}\right)^2,$$

$$\widehat{K} := \frac{1}{t_0 - n} \sum_{1<i\le n} \sum_{s\in T(i)} \left(X_{i,s}{}^{(1)} - \overline{X_\iota{}^{(1)}}\right)\left(X_{i,s}{}^{(2)} - \overline{X_\iota{}^{(2)}}\right),$$

$$\hat{L} := \frac{1}{n-1} [\sum_{1\le i\le n} \left(X_i{}^{(1)} - \overline{X^{(1)}}\right)\left(X_i{}^{(2)} - \overline{X^{(2)}}\right) - \frac{K}{n} \sum_{1\le i\le n} \frac{1}{t_i}].$$

where $\overline{X_\iota{}^{(\tau)}} := {}^1\!/_{t_i} \sum_{s\in T(i)} X_{i,s}{}^{(\tau)}$ and $\overline{X^{(\tau)}} := {}^1\!/_n \sum_{1\le i\le n} \overline{X_\iota{}^{(\tau)}}$.

(ii). For the parameters $B^{(\tau)}$, (τ =1, 2), only the positive part operator$[\cdot]^+$ prevents the following estimators from being unbiased:

$$\widehat{B}^{(\tau)} := \frac{1}{n-1} \left[\sum_{1\le i\le n} \left(\overline{X_\iota{}^{(\tau)}} - \overline{X^{(\tau)}}\right)^2 - \frac{\hat{A}^{(\tau)}}{n} \sum_{1\le i\le n} \frac{1}{t_i}\right]^+ . (18)$$

Proof.:
Let (τ =1, 2). Obviously, the linear estimator $\widehat{M}^{(\tau)}$ is unbiased.
That the estimator $\hat{A}^{(\tau)}$ is not biased follows from the fact that

$$E\left[\sum_{s\in T(i)} \left(X_{i,s}{}^{(\tau)} - \overline{X_\iota{}^{(\tau)}}\right)^2\right] = (t_i - 1)\hat{A}^{(\tau)}.$$

Similarly, $E\left[\sum_{s\in T(i)} \left(X_{i,s}{}^{(1)} - \overline{X_\iota{}^{(1)}}\right)\left(X_{i,s}{}^{(2)} - \overline{X_\iota{}^{(2)}}\right)\right] = (t_i - 1)K$ shows that $\widehat{K}$ is a no biased estimator. Next $\sum_i E\left[\left(\overline{X_\iota{}^{(1)}} - \overline{X^{(1)}}\right)\left(\overline{X_\iota{}^{(2)}} - \overline{X^{(2)}}\right)\right] = \frac{n-1}{n} \sum_i \frac{1}{t_i{}^2} \sum_{s,s'\in T(i)} Cov\left(X_{i,s}{}^{(1)}, X_{i,s}{}^{(2)}\right) = (n-1)[L + \frac{K}{n} \sum_i 1/t_i]$ shows that also the estimator $\widehat{L}$ is unbiased.
Finally,

$$E\left[\sum_i\left(\overline{X_\iota^{(\tau)}}-\overline{X^{(\tau)}}\right)^2\right]$$
$$=\frac{n-1}{n}\sum_i\frac{1}{{t_i}^2}\sum_{s,s'\in\mathrm{T(i)}}Cov\left({X_{i,s}}^{(\tau)},{X_{i,s}}^{(\tau)}\right)$$
$$=(n-1)[B^{(\tau)}+\frac{A^{(\tau)}}{n}\sum_i 1/t_i].$$

Gives the result in **(ii).**

5 The Bivariate Bühlmann-Straub model

This section provides a generalization of precedent section. The ideas are the same, when we look attentively at the assumptions of the previous model, we note that the hypothesis:

$Var\left[X^{(1)}{}_{i,s}|\Theta_\mathrm{i}\right] \coloneqq \alpha^{(1)}(\Theta_\mathrm{i})$ and $Var\left[X^{(2)}{}_{i,s}|\Theta_\mathrm{i}\right] \coloneqq \alpha^{(2)}(\Theta_\mathrm{i})$

We know that these conditional variances are descending with the exposure risk i. For this, the Bühlmann-Straub model generalizes the Bühlmann model by introduction of weight. We consider in this article an homogeneous insurance portfolio consisting of n≥2 contracts. Each contract i has been activated in the last t periods at least for one period. By T(i)⊆{1,…,t} we denote the set of the activated past periods of contract i, assuming that its cardinality t_i:=|T(i)|≥2. We set $t_0:=\sum_{1\le i\le n} t_i$. In order to be able to include the future period t+1, we set $T^+(i):=T(i)\cup\{t+1\}$. So, we consider same data of past model and we introduce the weights: $\varphi^{(\tau)}{}_{i,s}$ for $X^{(\tau)}{}_{i,s}$for all s∈T^+(i), i=1,…,n. Now we have for each contract i the set of pairs of random variables:

$$\left({X_i}^1,{X_i}^2\right)=\left({X_{i,s}}^1,{X_{i,s}}^2\right)_{s\in T^+(\mathrm{i})},$$

i =1,...,n. For example, ${X_{i,s}}^1 \; and \; {X_{i,s}}^2$ can be interpreted as respectively the amount and the number of claims of the contract i in the period s.

The random variables of the whole model are denoted by :

$$(X^1,X^2)=\left({X_i}^1,{X_i}^2\right)_{i=1,\dots,n},$$

We make the following model assumptions:

(BS1) The random structure variables $(\Theta_i)_{i=1,\dots,n}$ have the identical distribution $\mathbb{P}_\Theta$.

(BS2) The families of random variables $(\Theta_i, {X_i}^1, {X_i}^2)_{i=1,\dots,n}$, are mutually non-correlated with ${X_{i,s}}^1, {X_{i,s}}^2 \in L^2$ for all s ∈ T^+(i), i =1,...,n. The complete set of random variables $\{{X_{i,s}}^1, {X_{i,s}}^2$ i,s | s ∈ T^+(i),i =1,...,n} is assumed to be $\mathbb{P}$ -irreducible.

(BS3) For fixed i and conditioned by Θ_i, the pairs of random variables $\{{X_{i,s}}^1, {X_{i,s}}^2$ i,s | s ∈ T^+(i),i =1,...,n} are mutually non-correlated with the first and second conditional moments depending only on Θ_i [5]

τ =1, 2. Moreover, on a non-negligible set the pair $({X_{i,s}}^1, {X_{i,s}}^2)$is assumed to be $\mathbb{P}(\cdot|\Theta_i)$-irreducible:

$$\mathbb{P}_\Theta\left\{\omega\middle|{X_{i,s}}^1,{X_{i,s}}^2 \text{ is } \mathbb{P}(.|\Theta_\mathrm{i}(\omega))-\text{irreducible}\right\}>0. \quad (19)$$

For non-conditional moments (conditional variances and structure parameters) we use the same notation.

A first result from the assumptions made is the following [6]:

Proposition 5.1

Under the model assumptions **(BS1) - (BS3)** we have ∀i:

$$D_{i,1}:=\begin{pmatrix}B^{(1)}\otimes E_{t_i\times t_i}+A^{(1)}\otimes\Delta^{(1)}{}_\mathrm{i} & L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\dots,1)\\ L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\dots,1) & B^{(2)}\otimes E_{t_i\times t_i}+A^{(2)}\otimes\Delta^{(2)}{}_\mathrm{i}\end{pmatrix} \text{ Is invertible matrix, } \quad (20)$$

$$D_2 \coloneqq \frac{A^{(1)}}{\varphi^{(1)}{}_{i,s}}\frac{A^{(2)}}{\varphi^{(2)}{}_{i,s}}-K^2>0 \;\; \forall s \quad (21)$$

where $E_{t_i\times t_i}$ is matrix $t_i\times t_i$ where all components equal to 1, and $\Delta^{(\tau)}{}_\mathrm{i}:=\mathrm{diag}(1/$

$(\varphi^{(\tau)}{}_{i,s}))$ is a diagonal matrix, τ= 1,2 and s=1,..,t_i.

Proof

To show (20), we know by assumption **(BS1)** that the sequence of random variables ${X_{i,s}}^1, {X_{i,s}}^2$ is $\mathbb{P}(.\,|\Theta_\mathrm{i}(\omega)) -$ irreducible, and so is the pair vector $(\sum_{s\in T(i)} {X_{i,s}}^1, \sum_{s\in T(i)} {X_{i,s}}^2)$. Proposition 2 tells us that their covariance matrix is positive definite which is the statement in (20):

$$Cov\left(\sum_{s\in T(i)} {X_{i,s}}^1, \sum_{s\in T(i)} {X_{i,s}}^2\right) = \begin{pmatrix} B^{(1)} \otimes E_{t_i\times t_i} + A^{(1)} \otimes \Delta^{(1)}{}_\mathrm{i} & L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\dots,1) \\ L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\dots,1) & B^{(2)} \otimes E_{t_i\times t_i} + A^{(2)} \otimes \Delta^{(2)}{}_\mathrm{i} \end{pmatrix}.$$

For the inequality (21), the assumption **(BS3)** implies with Proposition that on a set of positive $\mathbb{P}_\Theta$ measure the covariance matrix with respect to the conditional probability $\mathbb{P}(.\,|\Theta_\mathrm{i}(\omega))$of the pair ${X_{i,s}}^1, {X_{i,s}}^2$ is positive definite:

$$Cov_{\mathbb{P}(\cdot|\Theta_i)}\left({X_{i,s}}^1, {X_{i,s}}^2\right) = \begin{pmatrix} \dfrac{\alpha^{(1)}(\Theta_i)}{\varphi^{(1)}{}_{i,s}} & \kappa(\Theta_i) \\ \kappa(\Theta_i) & \dfrac{\alpha^{(2)}(\Theta_i)}{\varphi^{(2)}{}_{i,s}} \end{pmatrix}.$$

This implies that for any pair $(\alpha_1, \alpha_1) \neq 0$ the conditional variance $Var_{\mathbb{P}(\cdot|\Theta_i)}\left(\alpha_1 {X_{i,s}}^1 + \alpha_2 {X_{i,s}}^2\right) \geq 0$ is strictly positive on a set of positive $\mathbb{P}_\Theta$ measure. Therefore also

$$E_\Theta\left[Var_{\mathbb{P}(\cdot|\Theta_i)}\left(\alpha_1 {X_{i,s}}^1 + \alpha_2 {X_{i,s}}^2\right)\right] = (\alpha_1 \quad \alpha_1)\begin{pmatrix} \dfrac{A^{(1)}}{\varphi^{(1)}{}_{i,s}} & K \\ K & \dfrac{A^{(2)}}{\varphi^{(2)}{}_{i,s}} \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} > 0$$

With real coefficients $\zeta_\mathrm{i}^{(\tau)} = \left(\zeta_{\mathrm{i},0}^{(\tau)}, \zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)}, \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)}\right)$.

Theorem5.1.

Under the assumptions **(BS1) -- (BS3)** the optimal solution to the linear credibility problem is:

$$\hat{\mu}^{(\tau)*}{}_i\,(\mathrm{X}^1{}_\mathrm{i}, \mathrm{X}^2{}_\mathrm{i})$$

$$= \left(1 - \zeta_\mathrm{i}{}^{(\tau,\tau)}\right)M^{(\tau)} + \frac{1}{\mathrm{t_i}}\sum_{\mathrm{s\in T(i)}}\left(\zeta_\mathrm{i}{}^{(\tau,\tau)}\ {X_{i,s}}^{(\tau)} + \zeta_\mathrm{i}{}^{(\tau,3-\tau)}\left({X_{i,s}}^{(3-\tau)} - M^{(3-\tau)}\right)\right)$$

τ =1, 2, where the credibility coefficients $\zeta_\mathrm{i}{}^{(\tau,\tau)}, \zeta_\mathrm{i}{}^{(\tau,3-\tau)}$ are given by the following formulas:

$$\begin{pmatrix}\zeta_\mathrm{i}{}^{(\tau,\tau)} \\ \zeta_\mathrm{i}{}^{(\tau,3-\tau)}\end{pmatrix} = \begin{pmatrix} B^{(\tau)} \otimes E_{t_i\times t_i} + A^{(\tau)} \otimes \Delta^{(\tau)}{}_\mathrm{i} & L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\dots,1) \\ L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\dots,1) & B^{(3-\tau)} \otimes E_{t_i\times t_i} + A^{(3-\tau)} \otimes \Delta^{(3-\tau)}{}_\mathrm{i} \end{pmatrix}\cdot\begin{pmatrix} B^{(\tau)} \otimes \mathrm{I} \\ L \otimes \mathrm{I}\end{pmatrix}$$

This shows that the matrix $\begin{pmatrix} \frac{A^{(1)}}{\varphi^{(1)}{}_{i,s}} & K \\ K & \frac{A^{(2)}}{\varphi^{(2)}{}_{i,s}} \end{pmatrix}$ is positive definite and consequently its determinant $D_2 := \frac{A^{(1)}}{\varphi^{(1)}{}_{i,s}}\frac{A^{(2)}}{\varphi^{(2)}{}_{i,s}} - K^2$is strictly positive.■

We want to find a linear credibility estimator of the bivariate system $(X_{i,t+1}{}^{1}, X_{i,t+1}{}^{2})$. Consequently, it is assumed that such estimators of $X_{i,t+1}{}^{(\tau)}$, τ =1, 2, have the form

$$\hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^{1},\mathrm{X}^{2}) = \zeta_{\mathrm{i},0}{}^{(\tau)} + \sum_{1<j<n}\sum_{\mathrm{s}\in\mathrm{T(j)}}(\zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)}\ X_{j,s}{}^{(\tau)} + \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)}\ X_{j,s}{}^{(3-\tau)})\ (16)$$

Where I is a vector (t_i × 1) with all composites equal to 1.

Remark 5.1.
In contrast to the standard credibility formula, here the credibility factors are not necessarily in[0, 1].

Proof.:
Obviously, the credibility problem

$$min_{\zeta_{\mathrm{i}}{}^{(1)},\zeta_{\mathrm{i}}{}^{(2)}} E\left[\sum_{\tau=1,2}\left(X_{i,t+1}{}^{(\tau)} - \hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^{1},\mathrm{X}^{2})\right)^{2}\right] = \sum_{\tau=1,2} min_{\zeta_{\mathrm{i}}{}^{(\tau)}}\ E[\left(X_{i,t+1}{}^{(\tau)} - \hat{\mu}^{(\tau)}{}_{i}\,(\mathrm{X}^{1},\mathrm{X}^{2})\right)^{2}].$$

Can be separated. As necessary conditions of optimality, we have for i,j ≤ n and s ∈ T(j):.

$$0 = \frac{\partial}{\partial\zeta_{\mathrm{i},0}{}^{(\tau)}} = E\left[X_{i,t+1}{}^{(\tau)} - \zeta_{\mathrm{i},0}{}^{(\tau)} - \sum_{1<j<n}\sum_{\mathrm{s}\in\mathrm{T(j)}}(\zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)}\ X_{j,s}{}^{(\tau)} + \zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)}\ X_{j,s}{}^{(3-\tau)})\right].$$

$$0 = \frac{\partial}{\partial\zeta_{\mathrm{i,j,s}}{}^{(\tau,\tau)}}\Psi = E\left[X_{j,s}{}^{(\tau)}(X_{i,t+1}{}^{(\tau)} - \zeta_{\mathrm{i},0}{}^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in\mathrm{T(j')}}(\zeta_{\mathrm{i,j',s'}}{}^{(\tau,\tau)}\ X_{j',s'}{}^{(\tau)} + \zeta_{\mathrm{i,j',s'}}{}^{(\tau,3-\tau)}\ X_{j',s'}{}^{(3-\tau)}))\right].$$

$$0 = \frac{\partial}{\partial\zeta_{\mathrm{i,j,s}}{}^{(\tau,3-\tau)}}\Psi = E\left[X_{j,s}{}^{(3-\tau)}(X_{i,t+1}{}^{(\tau)} - \zeta_{\mathrm{i},0}{}^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in\mathrm{T(j')}}(\zeta_{\mathrm{i,j',s'}}{}^{(\tau,\tau)}\ X_{j',s'}{}^{(\tau)} + \zeta_{\mathrm{i,j',s'}}{}^{(\tau,3-\tau)}\ X_{j',s'}{}^{(3-\tau)}))\right].$$

It follows that $\zeta_{\mathrm{i},0}{}^{(\tau)} = M^{(\tau)} - \sum_{1<j'<n}\sum_{s'\in\mathrm{T(j')}}(\zeta_{\mathrm{i,j',s'}}{}^{(\tau,\tau)}M^{(\tau)} + \zeta_{\mathrm{i,j',s'}}{}^{(\tau,3-\tau)}M^{(3-\tau)})$.

Replacing $\zeta_{\mathrm{i},0}{}^{(\tau)}$ in (22) it turns out that we can pass to centralized moments.

The last two lines in (22) are now equal to:

$$0=\delta_{i,j}B^{(\tau)}-\sum_{1<j'\le n}\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(B^{(\tau)}+\delta_{s,s'}\frac{A^{(\tau)}}{\varphi^{(\tau)}{}_{i,s}}\right)+\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(L+\delta_{s,s'}K\right)\right)$$

$$0=\delta_{i,j}B^{(\tau)}-\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(B^{(\tau)}+\delta_{s,s'}\frac{A^{(\tau)}}{\varphi^{(\tau)}{}_{i,s}}\right)+\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(L+\delta_{s,s'}K\right)\right),$$

$$0=\delta_{i,j}L-\sum_{1<j'\le n}\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(L+\delta_{s,s'}K\right)+\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(B^{(3-\tau)}+\delta_{s,s'}\frac{A^{(3-\tau)}}{\varphi^{(3-\tau)}{}_{i,s}}\right)\right)$$

$$\sum_{1<j'\le n}\sum_{s'\in T(j')}\delta_{j,j'}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(L+\delta_{s,s'}K\right)+\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(B^{(3-\tau)}+\delta_{s,s'}\frac{A^{(3-\tau)}}{\varphi^{(3-\tau)}{}_{i,s}}\right)\right).$$

$$0=\delta_{i,j}L-\sum_{s'\in T(j')}\left(\zeta_{i,j',s'}{}^{(\tau,\tau)}\left(L+\delta_{s,s'}K\right)+\zeta_{i,j',s'}{}^{(\tau,3-\tau)}\left(B^{(3-\tau)}+\delta_{s,s'}\frac{A^{(3-\tau)}}{\varphi^{(3-\tau)}{}_{i,s}}\right)\right)$$

Setting $\zeta_{i,j'}{}^{(\tau,3-\tau)}=\sum_{s'\in T(j')}\zeta_{i,j',s'}{}^{(\tau,3-\tau)}$ $\tau=1,2$, the summation over $s\in T(j)$ yields

$$\delta_{i,j}\,t_jB^{(\tau)}=\zeta_{i,j}{}^{(\tau,\tau)}\left(t_jB^{(\tau)}+\frac{A^{(\tau)}}{\varphi^{(\tau)}{}_{i,s}}\right)+\zeta_{i,j}{}^{(\tau,3-\tau)}\left(t_jL+K\right)$$

$$\delta_{i,j}\,t_jL=\zeta_{i,j}{}^{(\tau,\tau)}\left(t_jL+K\right)+\zeta_{i,j}{}^{(\tau,3-\tau)}\left(t_jB^{(3-\tau)}+\frac{A^{(3-\tau)}}{\varphi^{(3-\tau)}{}_{i,s}}\right)$$

By (14) the determinant of the linear system is positive such that $0=\zeta_{i,j}{}^{(\tau,\tau)}=\zeta_{i,j}{}^{(\tau,3-\tau)}$ for i $\neq$ j and Hence we obtain the linear system[6]

$$\begin{pmatrix}\sum_{s\in T(i)}\zeta_i{}^{(\tau,\tau)}\\ \sum_{s\in T(i)}\zeta_i{}^{(\tau,3-\tau)}\end{pmatrix}=\frac{1}{\Xi^{(\tau)}{}_i}\begin{pmatrix}t_iB^{(\tau)}\left(t_iB^{(\tau)}+A^{(\tau)}\right)-t_iL\left(t_iL+K\right)\\ -t_iB^{(\tau)}\left(t_iL+K\right)+t_iL\left(t_iB^{(3-\tau)}+A^{(3-\tau)}\right)\end{pmatrix}^{-1}$$

$$\begin{pmatrix}B^{(\tau)}\otimes I\\ L\otimes I\end{pmatrix}=\begin{pmatrix}B^{(\tau)}\otimes E_{t_i\times t_i}+A^{(\tau)}\otimes\Delta^{(\tau)}{}_i & L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\ldots,1)\\ L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\ldots,1) & B^{(3-\tau)}\otimes E_{t_i\times t_i}+A^{(3-\tau)}\otimes\Delta^{(3-\tau)}{}_i\end{pmatrix}\cdot\begin{pmatrix}\zeta_i{}^{(\tau,\tau)}\\ \zeta_i{}^{(\tau,3-\tau)}\end{pmatrix}$$

So,

$$\begin{pmatrix}\zeta_i{}^{(\tau,\tau)}\\ \zeta_i{}^{(\tau,3-\tau)}\end{pmatrix}=\begin{pmatrix}B^{(\tau)}\otimes E_{t_i\times t_i}+A^{(\tau)}\otimes\Delta^{(\tau)}{}_i & L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\ldots,1)\\ L\otimes E_{t_i\times t_i}+K\otimes\mathrm{diag}(1,\ldots,1) & B^{(3-\tau)}\otimes E_{t_i\times t_i}+A^{(3-\tau)}\otimes\Delta^{(3-\tau)}{}_i\end{pmatrix}^{-1}\cdot\begin{pmatrix}B^{(\tau)}\otimes I\\ L\otimes I\end{pmatrix}\quad(23)$$

Remark5.2
To return from bivariate Bühlmann-Straub credibility estimators to bivariate Bühlmann credibility estimators, we put $\varphi^{(\tau)}{}_{i,1}=$

$\varphi^{(\tau)}{}_{i,2} = \cdots = \varphi^{(3-\tau)}{}_{i,1} = \varphi^{(3-\tau)}{}_{i,2} = .. = 1$ we have the Covariance matrix equal to:

where $\Xi^{(\tau)}{}_{\mathrm{i}} \coloneqq \left(t_i B^{(\tau)} + A^{(\tau)}\right)\left(t_i B^{(3-\tau)} + A^{(3-\tau)}\right) - (t_i L + K)^2$

$$Cov\left(\sum_{s\in T(i)} X_{i,s}{}^{1}, \sum_{s\in T(i)} X_{i,s}{}^{2}\right) = \begin{pmatrix} B^{(1)} \otimes E_{t_i\times t_i} + A^{(1)} \otimes \Delta^{(1)}{}_{\mathrm{i}} & L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\ldots,1) \\ L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\ldots,1) & B^{(2)} \otimes E_{t_i\times t_i} + A^{(2)} \otimes \Delta^{(2)}{}_{\mathrm{i}} \end{pmatrix}$$

So, the system (23) becomes:

$$\begin{pmatrix} \zeta_{\mathrm{i}}{}^{(\tau,\tau)} \\ \zeta_{\mathrm{i}}{}^{(\tau,3-\tau)} \end{pmatrix} = \begin{pmatrix} B^{(\tau)} \otimes E_{t_i\times t_i} + A^{(\tau)} \otimes \mathrm{diag}(1,\ldots,1) & L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\ldots,1) \\ L \otimes E_{t_i\times t_i} + K \otimes \mathrm{diag}(1,\ldots,1) & B^{(3-\tau)} \otimes E_{t_i\times t_i} + A^{(3-\tau)} \otimes \mathrm{diag}(1,\ldots,1) \end{pmatrix}^{-1} \cdot \begin{pmatrix} B^{(\tau)} \otimes \mathrm{I} \\ L \otimes \mathrm{I} \end{pmatrix}$$

When applying the sum of s, We have

$$\begin{pmatrix} \sum_{s\in T(i)} \zeta_{\mathrm{i}}{}^{(\tau,\tau)} \\ \sum_{s\in T(i)} \zeta_{\mathrm{i}}{}^{(\tau,3-\tau)} \end{pmatrix} = \begin{pmatrix} t_i B^{(\tau)} + A^{(\tau)} & t_i L + K \\ t_i L + K & t_i B^{(3-\tau)} + A^{(3-\tau)} \end{pmatrix}^{-1} \cdot \begin{pmatrix} t_i B^{(\tau)} \\ t_i L \end{pmatrix}$$

or,

.

Now, the system can be rewritten as : $\begin{pmatrix} A^{(\tau)} & K \\ K & A^{(3-\tau)} \end{pmatrix} \cdot \begin{pmatrix} \zeta_{\mathrm{i,s'}}{}^{(\tau,\tau)} \\ \zeta_{\mathrm{i,s'}}{}^{(\tau,3-\tau)} \end{pmatrix} = \begin{pmatrix} \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)B^{(\tau)} - \zeta_{\mathrm{i}}{}^{(3-\tau)}\mathrm{L} \\ \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)\mathrm{L} - \zeta_{\mathrm{i}}{}^{(3-\tau)}B^{(\tau)} \end{pmatrix}$,

$$\begin{pmatrix} \zeta_{\mathrm{i,s'}}{}^{(\tau,\tau)} \\ \zeta_{\mathrm{i,s'}}{}^{(\tau,3-\tau)} \end{pmatrix} = \begin{pmatrix} A^{(\tau)} & K \\ K & A^{(3-\tau)} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)B^{(\tau)} - \zeta_{\mathrm{i}}{}^{(3-\tau)}\mathrm{L} \\ \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)\mathrm{L} - \zeta_{\mathrm{i}}{}^{(3-\tau)}B^{(\tau)} \end{pmatrix}$$

$$\begin{pmatrix} \zeta_{\mathrm{i,s'}}{}^{(\tau,\tau)} \\ \zeta_{\mathrm{i,s'}}{}^{(\tau,3-\tau)} \end{pmatrix} = \frac{1}{A^{(\tau)}A^{(3-\tau)} - K^2} \begin{pmatrix} A^{(3-\tau)} & -K \\ -K & A^{(\tau)} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)B^{(\tau)} - \zeta_{\mathrm{i}}{}^{(3-\tau)}\mathrm{L} \\ \left(1-\zeta_{\mathrm{i}}{}^{(\tau)}\right)\mathrm{L} - \zeta_{\mathrm{i}}{}^{(3-\tau)}B^{(\tau)} \end{pmatrix}$$

Finally, we have the bivariate credibility estimators of Bühlmann:

$$\begin{pmatrix} \zeta_{\mathrm{i,s'}}{}^{(\tau,\tau)} \\ \zeta_{\mathrm{i,s'}}{}^{(\tau,3-\tau)} \end{pmatrix} = \begin{pmatrix} \dfrac{\left(t_i B^{(\tau)} + A^{(\tau)}\right)B^{(\tau)} - \mathrm{L}(t_i L + K)}{\Xi^{(\tau)}{}_{\mathrm{i}}} \\ \dfrac{-B^{(\tau)}(t_i L + K) + L\left(t_i B^{(3-\tau)} + A^{(3-\tau)}\right)}{\Xi^{(\tau)}{}_{\mathrm{i}}} \end{pmatrix}.$$

Theorem5.2.

(i). For the parameters: M , A , K, and L, (τ =1, 2), the following respective estimators are unbiased[10],[9]:

$\widehat{M}^{(\tau)} \coloneqq \frac{1}{t_0} \sum_{1<\mathrm{i}\leq \mathrm{n}} \sum_{\mathrm{s}\in \mathrm{T(i)}} X_{i,s}{}^{(\tau)}$,

$$\hat{A}^{(\tau)} \coloneqq \frac{1}{t_0 - n} \sum_{1<\mathrm{i}\leq \mathrm{n}} \sum_{\mathrm{s}\in \mathrm{T(i)}} \varphi^{(\tau)}{}_{i,s} \left(X_{i,s}{}^{(\tau)} - X_{i,\varphi}{}^{(\tau)}\right)^2,$$

$$\widehat{K} \coloneqq \frac{1}{c_1}\sum_{1<\mathrm{i}\le\mathrm{n}}\sum_{\mathrm{s}\in\mathrm{T(i)}}\left(X_{i,s}{}^{(\tau)} - X_{i,\varphi}{}^{(\tau)}\right)\left(X_{i,s}{}^{(3-\tau)} - X_{i,\varphi}{}^{(3-\tau)}\right),$$

$$\hat{L} \coloneqq \frac{1}{n-2+n\sum_{1<\mathrm{i}\le\mathrm{n}}\frac{\varphi^{(\tau)}{}_{\mathrm{i},.}\varphi^{(3-\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}\varphi^{(3-\tau)}{}_{.,.}}}\Big[\sum_{1\le\mathrm{i}\le\mathrm{n}}\left(X_{i,\varphi}{}^{(\tau)} - X_{\varphi,\varphi}{}^{(\tau)}\right)\left(X_{i,\varphi}{}^{(3-\tau)} - X_{\varphi,\varphi}{}^{(3-\tau)}\right) - K.(c_3\Big].$$

(ii). For the parameters $B^{(\tau)}$, (τ =1, 2), only the positive part operator $[\cdot]^{+}$ prevents the following estimators from being unbiased:

$$\hat{B}^{(\tau)} \coloneqq c_2\Big[\frac{n}{n-1}\sum_{1\le\mathrm{i}\le\mathrm{n}}\frac{\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}}\left(X_{i,\varphi}{}^{(\tau)} - X_{\varphi,\varphi}{}^{(\tau)}\right)^2 - n\frac{\hat{A}^{(\tau)}}{\varphi^{(\tau)}{}_{.,.}}\Big]^{+}$$

where $\varphi^{(\tau)}{}_{\mathrm{i},.} := \sum_s \varphi^{(\tau)}{}_{\mathrm{i,s}}$

$\varphi^{(\tau)}{}_{.,.} := \sum_i \varphi^{(\tau)}{}_{\mathrm{i},.}$

$$X_{i,\varphi}{}^{(\tau)} := \sum_s \frac{\varphi^{(\tau)}{}_{\mathrm{i,s}}}{\varphi^{(\tau)}{}_{\mathrm{i},.}} X_{i,s}{}^{(\tau)}$$

$$X_{\varphi,\varphi}{}^{(\tau)} := \sum_s \frac{\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}} X_{i,s}{}^{(\tau)}$$

$$c_1 \coloneqq t_0 - 2n - \sum_{\mathrm{i,s}} \mathrm{t_i}\frac{\varphi^{(\tau)}{}_{\mathrm{i,s}}\varphi^{(3-\tau)}{}_{\mathrm{i,s}}}{\varphi^{(\tau)}{}_{\mathrm{i},.}\varphi^{(3-\tau)}{}_{\mathrm{i},.}}$$

$$c_3 \coloneqq \sum_{\mathrm{i,s}} \varphi^{(\tau)}{}_{\mathrm{i,s}}\varphi^{(3-\tau)}{}_{\mathrm{i,s}}\Big(\frac{1}{\varphi^{(\tau)}{}_{\mathrm{i},.}\varphi^{(3-\tau)}{}_{\mathrm{i},.}} - \frac{1}{\varphi^{(\tau)}{}_{\mathrm{i},.}\varphi^{(3-\tau)}{}_{.,.}} - \frac{1}{\varphi^{(\tau)}{}_{.,.}\varphi^{(3-\tau)}{}_{\mathrm{i},.}} + \frac{\mathrm{n}}{\varphi^{(\tau)}{}_{.,.}\varphi^{(3-\tau)}{}_{.,.}}\Big)$$

And $c_2 \coloneqq \frac{n-1}{n}\big(\sum_{\mathrm{i}}\frac{\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}}\big(1 - \frac{\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}}\big)\big)^{-1}$.

Proof. :

Let(τ =1, 2). Obviously, the linear estimator $\widehat{M}^{(\tau)}$ is unbiased. That the estimator $\hat{A}^{(\tau)}$is not biased follows from the fact that

$$E\left[\sum_{i,s}\varphi^{(\tau)}{}_{\mathrm{i,s}}\left(X_{i,s}{}^{(\tau)} - X_{i,\varphi}{}^{(\tau)}\right)^2\right] = (t_0 - n)\hat{A}^{(\tau)}.$$

Similarly, $E\left[\sum_{i,s}\left(X_{i,s}{}^{(\tau)} - X_{i,\varphi}{}^{(\tau)}\right)\left(X_{i,s}{}^{(3-\tau)} - X_{i,\varphi}{}^{(3-\tau)}\right)\right] = c_1 K.$ Shows that $\widehat{K}$ is a non-biased estimator. Next,

$$\sum_i E\left[\left(X_{i,\varphi}{}^{(\tau)} - X_{\varphi,\varphi}{}^{(\tau)}\right)\left(X_{i,\varphi}{}^{(3-\tau)} - X_{\varphi,\varphi}{}^{(3-\tau)}\right)\right]$$

$$= \sum_i \left[L\left[1 - \frac{\varphi^{(3-\tau)}{}_{\mathrm{i},.}}{\varphi^{(3-\tau)}{}_{.,.}} - \frac{\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(\tau)}{}_{.,.}} + \sum_i \frac{\varphi^{(3-\tau)}{}_{\mathrm{i},.}\varphi^{(\tau)}{}_{\mathrm{i},.}}{\varphi^{(3-\tau)}{}_{.,.}\varphi^{(\tau)}{}_{.,.}}\right] + K\left[\sum_s \frac{\varphi^{(3-\tau)}{}_{\mathrm{i,s}}\varphi^{(\tau)}{}_{\mathrm{i,s}}}{\varphi^{(3-\tau)}{}_{\mathrm{i},.}\varphi^{(\tau)}{}_{\mathrm{i},.}} - \sum_s \frac{\varphi^{(3-\tau)}{}_{\mathrm{i,s}}\varphi^{(\tau)}{}_{\mathrm{i,s}}}{\varphi^{(3-\tau)}{}_{.,.}\varphi^{(\tau)}{}_{\mathrm{i},.}} - \sum_s \frac{\varphi^{(3-\tau)}{}_{\mathrm{i,s}}\varphi^{(\tau)}{}_{\mathrm{i,s}}}{\varphi^{(\tau)}{}_{.,.}\varphi^{(3-\tau)}{}_{\mathrm{i},.}} + \sum_{i,s} \frac{\varphi^{(3-\tau)}{}_{\mathrm{i,s}}\varphi^{(\tau)}{}_{\mathrm{i,s}}}{\varphi^{(\tau)}{}_{.,.}\varphi^{(3-\tau)}{}_{.,.}}\right]\right].$$

Shows that also the estimator $\hat{L}$ is unbiased. Finally,

$$E\left[\sum_i \frac{\varphi^{(\tau)}{}_{i,.}}{\varphi^{(\tau)}{}_{.,.}}\left(X_{i,\varphi}{}^{(\tau)} - X_{\varphi,\varphi}{}^{(\tau)}\right)^2\right] = B^{(\tau)}\left(\frac{\varphi^{(\tau)}{}_{.,.}{}^{2} - \sum_i \varphi^{(\tau)}{}_{i,.}{}^{2}}{\varphi^{(\tau)}{}_{.,.}{}^{2}}\right) + (n-1)\frac{A^{(\tau)}}{\varphi^{(\tau)}{}_{.,.}}$$

We can observe from the table above, that the MSE and RMSE of the bivarite Bühlmann estimators $\mu^1(z)$ are consistent smaller than classic Bühlmann estimators $\mu^1(z)$. In the sense mean square error, the estimators of the bivarite Bühlmann $\mu^1(z)$ are consistently better than the existing classic Bühlmann estimators with $X^{(1)}$.
Gives the result in (ii).

6 Example

In this section, we present a numerical example to illustrate the bivarite Bühlmann credibility estimators given in section 3 and compare these estimators with the existing results in [1] and [2](classic Bühlmann estimator). In this simulation, we assume that the claims $X_{ij}^2 \sim \text{lognormal}(-0.6,1)$ *and* $X_{ij}^1 \sim \text{poisson}(\text{lambda} = 3)$ for $i = 1,\dots,100$ *and* $j = 1,\dots,6$ respectivly. To compare the bivariate credibility premiums, we simulated six years of experience for portfolios of 100 contracts. We calculated the bivariate credibility premium using the first five years of experience and compared it with the actual outcome, X_{i6}^2. [8]For each simulation,(Monte Carlo) the accuracy of the various formulas was measured by the mean square error $MSE = \frac{1}{100}\sum_{i=1}^{100}\left(\mu_{i6}^{(\tau)} - X_{i6}^{(\tau)}\right)^2, for\ \tau = 1,2.$
And the relative mean square error: $RMSE = \frac{1}{100}\sum_{i=1}^{100}\left(\frac{\mu_{i6}^{(\tau)} - X_{i6}^{(\tau)}}{\mu_{i6}^{(\tau)}}\right)^2, for\ \tau = 1,2.$
These errors were then averaged over 5,000 simulations. We also recorded the number of times each formula had the smallest MSE and RMSE.
The calculations were coded in R(R Development CoreTeam2005).The functions used to simulate the data and compute the structure parameters are part of the R package *actuar*. The table 1 presents the results when structure parameters were estimated using the estimators of section

Table1. Results of 5000 simulations

	MSE	RMSE
Classic Bühlmann		
$\mu^1(z)$	2.980076	1.72629
$\mu^2(z)$	0.8180612	0.9044673
$\mu^{2/1}(z)$	0.3269737	0.5718162
Bivarite Bühlmann		
$\mu^1(z)$	1.280576	1.131626
$\mu^2(z)$	4.231229	2.056995
$\mu^{2/1}(z)$	1.992162	1.41144

7 Simulations

R code
simulation of Bühlmann classic X2

```
 Appel package actuar
nsimul<-5000
prime<- matrix(0, nrow=100, ncol=nsimul)
for(i in 1 : nsimul){ X2<-  matrix(c(rlnorm(600,-0.6,1)),ncol=6 ) #matrice de montant de
sinistre
```

```
X2<-matrix(X2,nrow=100,ncol=5)
x<-data.frame(contract=1:100, matrix(X2,nrow=100))
fit<-cm(~contract,x)
prime[,i]<-predict(fit)
}
Prime.buhlmann<-matrix(0,nrow=100,ncol=1)
for(i in 1: 100){ Prime.buhlmann [i,]<-mean(prime[i,])}

data <- data.frame(actual=X2[,6],
predicted= Prime.buhlmann)
mean((data$actual - data$predicted)^2)

sqrt(mean((data$actual - data$predicted)^2))
```

simulation of classic Bühlmann X2/X1

```
 Appel package actuar
n=100
nsimul<-5000
prime<- matrix(0, nrow=100, ncol=nsimul)
for(i in 1 : nsimul){ X2<-  matrix(c(rlnorm(600,-0.6,1)),ncol=6 ) #matrice de montant de
sinistre
X2<-matrix(X2,nrow=100,ncol=5)
X1<-  matrix(c(rpois(600,lambda=3)),ncol=6 ) #matrice de nombre de sinistre

  X1<-matrix(X1,nrow=100,ncol=5)
for( i in 1:100){for(j in 1:5){ if( X1[i,j]==0){X1[i,j]<-1}}}
for( i in 1:100){for(j in 1:6){ if( X1[i,j]==0){X1[i,j]<-1}}}

x<-data.frame(contract=1:100, matrix(X2/X1,nrow=100))
fit<-cm(~contract,x)
prime[,i]<-predict(fit)
                                                            }
Prime.buhlmann<-matrix(0,nrow=100,ncol=1)
for(i in 1: 100){ Prime.buhlmann [i,]<-mean(prime[i,])}
p<-(X2/X1)[,6]
data <- data.frame(actual=p,
predicted= Prime.buhlmann)
mean((data$actual - data$predicted)^2)

sqrt(mean((data$actual - data$predicted)^2))
```

simulation of classic Bühlmann X1

```
 Appel package actuar
nsimul<-5000
prime<- matrix(0, nrow=100, ncol=nsimul)
for(i in 1 : nsimul){ X2<-  matrix(c(rlnorm(600,-0.6,1)),ncol=6 ) #matrice de montant de
sinistre
X2<-matrix(X2,nrow=100,ncol=5)
X1<-  matrix(c(rpois(600,lambda=3)),ncol=6 ) #matrice de nombre de sinistre

  X1<-matrix(X1,nrow=100,ncol=5)
```

```
x<-data.frame(contract=1:100, matrix(X1,nrow=100))
fit<-cm(~contract,x)
prime[,i]<-predict(fit)
}
Prime.buhlmann<-matrix(0,nrow=100,ncol=1)
for(i in 1: 100){ Prime.buhlmann [i,]<-mean(prime[i,])}
p<- X1[,6]
data <- data.frame(actual=p,
predicted= Prime.buhlmann)
mean((data$actual - data$predicted)^2)

sqrt(mean((data$actual - data$predicted)^2))
```

Simulation of bivariate Bühlmann model

```
n=100
Fact.credib11<-1 :n ;  Fact.credib12<-1 :n ;  Fact.credib22<-1 :n ; Fact.credib21<-1 :n ;
Prime.credib1 <-1:n ; Prime.credib2<- 1:n  ;  C<-rep(1, 6) ; Prime.credib.bivar<- 1:n
  S<-matrix(1, ncol=5,nrow=100)
nsimul<-5000
prime<- matrix(0, nrow=100, ncol=nsimul)
for(j in 1 : nsimul){
X1<-  matrix(c(rpois(600,lambda=3)),ncol=6 ) #matrice de nombre de sinistre
X2<-  matrix(c(rlnorm(600,-0.6,1)),ncol=6 ) #matrice de montant de sinistre
X2<-matrix(X2,nrow=100,ncol=5)
  X1<-matrix(X1,nrow=100,ncol=5)
for( i in 1:100){for(j in 1:5){ if( X1[i,j]==0){X1[i,j]<-1}}}


  X1.<-1:n;  X2.<-1:n;  i<-1:n; E=matrix(1, ncol=5, nrow=n)
   for(i in 1:n){X1.[i]=(1/sum(S[i,]))*sum(X1[i,])}

   X1..=(1/n)*sum(X1.)

   for(i in 1:n){X2.[i]=(1/ sum(S[i,]))*sum(X2[i,])}

   X2..=(1/n)*sum(X2.)

  a1=(1/(sum(S)-n))*(sum( (X1-(X1.*E))^2))

      a2=(1/(sum(S)-n))*(sum( (X2-(X2.*E))^2))

   K=(1/(sum(S)-n))*sum((X1-(X1.*E))*(X2-(X2.*E)))


   L=(1/(n-1))*(sum((X1.-X1..)*(X2.-X2..))-(K/n)*(sum(1/sum(S[i,]))))


   B1= abs((1/(n-1))*(sum((X1.-X1..)^2)-(a1/n)*(sum(1/sum(S[i,])))));  B2 = abs((1/(n-
1))*(sum((X2.-X2..)^2)-(a2/n)*(sum(1/sum(S[i,])))))
```

```
  for(i in 1: n){  (Fact.credib22[i]=(sum(S[i,])/( (sum(S[i,])*B2+a2)*(sum(S[i,])*B1+a1)-
(sum(S[i,])*L+K))*(B2*(sum(S[i,])*B1+a1)-L*(sum(S[i,])*L+K))) )}

   for(i in 1: n){ (Fact.credib21[i]=(sum(S[i,])/( (sum(S[i,])*B2+a2)*(sum(S[i,])*B1+a1)-
(sum(S[i,])*L+K))*(a2*L+B2*K)) ) }


  for(i in 1: n){  (Prime.credib2[i]=(1-
Fact.credib22[i])*X2..+(1/sum(S[i,]))*sum(Fact.credib22[i]*X2[i,]+Fact.credib21[i]*(X1[i,]-
X1..*C)))   }
for(i in 1: n){  (Prime.credib.bivar [i]= Prime.credib2[i]/ Prime.credib1[i])}

prime[,j]<- Prime.credib1
}
Prime.buhlmann<-matrix(0,nrow=100,ncol=1)
for(i in 1: 100){ Prime.buhlmann [i,]<-mean(prime[i,])}


p<- X2[,6]
data <- data.frame(actual=p,
predicted= Prime.buhlmann)
mean((data$actual - data$predicted)^2)

sqrt(mean((data$actual - data$predicted)^2))
```

8 Conclusions

Insurance and more precisely its branch the theory of credibility plays a key role of economic stabilizer for households in times of crisis, because it allows to smooth the consumption of individuals facing shocks, whether they are individual or collective in nature such as natural disasters or financial crises. For example, auto insurance and fire insurance are products that perform this stabilizing function. Another example, the number of catastrophic events and the amount of economic losses is varying in different world regions.

So credibility theory is also a stable source of finance for financial markets and for the economy, because it promotes credit and investment in a long-term perspective. The paper describes a generalization a univariate Bühlmann model, *by using additional sources of data,* where the new credibility is derived.

In this article, the bivariate credibility model is refined by the introduction of the irreducible random variables. This concept is very important in the calculation of bivariate credibility premium which helps us ensure the covariance matrix inversion. Thus, the future prospects are Quadratic Bivariate Credibility.

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