Exact Fill Rates for the (R, S) Inventory Control with Discrete Distributed Demands for the Backordering Case

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The fill rate is usually computed by using the traditional approach, which calculates it as the complement of the quotient between the expected unfulfilled demand and the expected demand per replenishment cycle, instead of directly the expected fraction of fulfilled demand. Furthermore the available methods to estimate the fill rate apply only under specific demand conditions. This paper shows the research gap regarding the estimation procedures to compute the fill rate and suggests: (i) a new exact procedure to compute the traditional approximation for any discrete demand distribution; and (ii) a new method to compute the fill rate directly as the fraction of fulfilled demand for any discrete demand distribution. Simulation results show that the latter methods outperform the traditional approach, which underestimates the simulated fill rate, over different demand patterns. This paper focuses on the traditional periodic review, base stock system when backlogged demands are allowed.
Keywords: *Inventory, Fill Rate, Periodic Review, Backordering, Discrete Demand*

1 Introduction

The traditional problem of the periodic review, base stock (*R, S*) system is usually on the determination of the base stock, *S*, such that total costs are minimized or some target customer service level is fulfilled. Even if the cost criterion is used for that purpose, the service level is usually included by imposing penalty costs on shortages [1] or by using it to compute the base stock in order to minimize holding costs of the system [2]. However in real situations these costs are difficult to know and estimate. Particularly difficult is to measure the costs incurred by having insufficient stock to attempt the demand since they include such factors as loss of customers' goodwill [3] [4]. For this reason, practitioners use the service level criterion to establish the base stock. Therefore accurate expressions to estimate customer service levels are required. Appropriate service indicators are the cycle service level and the fill rate, being the latter the most used in practice since it considers not only the possibility that the system is out of stock [1], but also the size of the unfulfilled demand when it occurs [5] [6].

This paper focuses on the exact estimation of the fill rate in (*R, S*) systems. Furthermore, when managing inventories it is required to know how to proceed when an item is out of stock and a customer order arrives. There are two extreme cases: the backordering case (=any unfulfilled demand is backordered and filled as soon as possible); and the lost sales case (=any unfulfilled demand is lost). This paper focuses on the backordering case.

The fill rate is defined as the fraction of demand that is immediately fulfilled from on hand stock [7]. Common approach to estimate it consists on computing the number of units short, i.e. the demand that is not satisfied, instead of computing directly the fulfilled demand per replenishment cycle. This approach, known in the related literature as the traditional approximation and denoted by β_{Trad} further on, consists of calculating the complement of the quotient between the expected unfulfilled demand per replenishment cycle (also known as expected shortage) and the total expected demand per replenishment cycle as follows:

$$\beta_{Trad} = 1 - \frac{E\left(\text{unfulfilled demand per replenishment cycle}\right)}{E\left(\text{total demand per replenishment cycle}\right)} \quad (1)$$

One limitation of the available methods devoted to estimating β_{Trad} in the (*R, S*) system for the backordering case is that they estimate it only for specific demand conditions.

In this sense, [3], [8], [9] and [10] suggest methods to estimate it when demand is normal distributed whereas [11], [12], [13] when demand follows any continuous distribution. When demand is discrete, only [3] suggest a method to estimate β_{Trad} for Poisson demands but to the rest of our knowledge no methods are available to estimate it when demand follows any discrete distribution function and backlog demands are allowed.

Another approach to compute the fill rate consists of directly estimating the fraction of the fulfilled demand per replenishment cycle instead of determining the expected shortage, as follows:

$$\beta = E\left(\frac{\text{fulfilled demand per replenishment cycle}}{\text{total demand per replenishment cycle}}\right) \quad (2)$$

However, expression (1) and expression (2) are not equivalent. Note that if X and Y are independent random variables it is true that $E(X \cdot Y) = E(X) \cdot E(Y)$ and, analogously, $E\left(\frac{X}{Y}\right) = E(X) \cdot E\left(\frac{1}{Y}\right)$.

However $E\left(\frac{1}{Y}\right) \neq \frac{1}{E(Y)}$ and therefore $E\left(\frac{X}{Y}\right) \neq \frac{E(X)}{E(Y)}$ (see for example [14]).

Thus, applying this reasoning to the definition of the fill we find that

$$E\left(\frac{\text{fulfilled demand per replenishment cycle}}{\text{total demand per replenishment cycle}}\right) \neq \frac{E(\text{fulfilled demand per replenishment cycle})}{E(\text{total demand per replenishment cycle})}$$

Since

$$\frac{E(\text{fulfilled demand per replenishment cycle})}{E(\text{total demand per replenishment cycle})} = 1 - \frac{E(\text{unfulfilled demand per replenishment cycle})}{E(\text{total demand per replenishment cycle})}$$

then

$$E\left(\frac{\text{fulfilled demand per replenishment cycle}}{\text{total demand per replenishment cycle}}\right) \neq 1 - \frac{E(\text{unfulfilled demand per replenishment cycle})}{E(\text{total demand per replenishment cycle})}$$

[15] proposes methods to estimate both expression (1) and (2) for any discrete demand pattern when inventory is managed following the lost sales case principle. However, there is not available any method to estimate β_{Trad} and β when demand is modeled by any discrete distribution and the inventory is managed following the backordering case, i.e. when unfulfilled demand is backlogged to the following cycle. This paper fulfills this research gap and suggests two new and exact methods to estimate both expressions (Section 3). Furthermore, we present and discuss some illustrative examples of the performance of both versus a simulated fill rate and over different demand patterns (Section 4). The discussion and summary of this work are summarized in Section 5.

2 Basic Notation and Assumptions

The traditional periodic review, base stock (R, S) system places replenishment orders every R units of time of sufficient magnitude to raise the inventory position to the base stock S. The replenishment order is received L periods after being launched. Figure 1 shows an example of the evolution of the on hand stock (= stock that is physically on shelf), the net stock (= on hand stock - backorders) and the inventory position (= on hand stock + stock on order – backorders) for the backordering case.

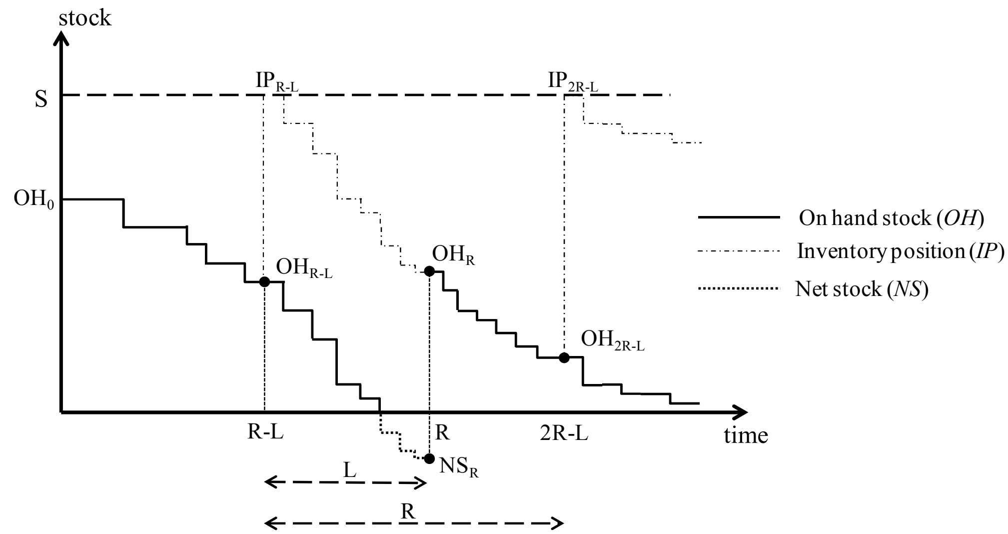

Fig. 1. Example of the evolution of a (*R, S*) system (backordering case)

Notation in the Figure 1 and in the rest of the paper is as follows:

S	=	base stock (units),
R	=	review period corresponding to the time between two consecutive reviews and replenishment cycle corresponding to the time between two consecutive deliveries (time units),
L	=	lead time for the replenishment order (time units),
OH_t	=	on hand stock at time t (units),
IP_t	=	inventory position at time t (units),
NS_t	=	net stock at time t (units),
D_t	=	total demand during t consecutive periods (units),
$f_t(\cdot)$	=	probability mass function of D_t,
$F_t(\cdot)$	=	cumulative distribution function of D_t,

This paper assumes that: (i) time is discrete and is organized in a numerable and infinite succession of equi-spaced instants; (ii) the lead time, L, is constant; (iii) the replenishment order is added to the inventory at the end of the period in which it is received, hence these products are available for the next period; (iv) demand during a period is fulfilled with the on hand stock at the beginning of the period; and (v) demand process is discrete, stationary and i.i.d.

3 Estimation of the Fill Rate in a Discrete Demand Context

3.1 Derivation of an Exact Method to Compute β_{Trad}

The traditional approximation of the fill rate computes the complement of the ratio between the expected unfulfilled demand (expected shortage) and the expected demand per replenishment cycle as shown in expression (1). The expected demand can be straightforwardly computed so all that is left to compute is the expected unfulfilled demand per replenishment cycle. Then, if at the beginning of the cycle there is not stock on shelf to satisfy any demand, the net stock at this time is zero or negative ($NS_0 \leq 0$) and therefore the expected shortage is equal to the expected demand during the replenishment cycle. Hence, the β_{Trad} is equal to zero.

On the other hand if the net stock at the beginning of the cycle is positive ($NS_0 > 0$), the shortage is equal to the difference between the NS_0 and the amount of demand that exceed NS_0 during that cycle. By definition, the net stock when positive can be from 1 to S, and hence:

$$E\left(\text{unfulfilled demand per replenishment cycle}\right)=\sum_{NS_0=1}^{S}P\left(NS_0\right)\cdot\sum_{D_R=NS_0+1}^{\infty}\left(D_R-NS_0\right)P\left(D_R\right) \quad (3)$$

Since the net stock is equivalent to the inventory position minus the on order stock, the net stock balance at the beginning of the cycle is:

$NS_0 = S - D_L$.

Then,

$P\left(NS_0\right)=P\left(D_L=S-NS_0\right)=f_L\left(S-NS_0\right)$.

Therefore, β_{Trad} when demand follows any discrete distribution function can be estimated with the following expression:

$$\beta_{Trad}=1-\frac{\sum_{NS_0=1}^{S}f_L\left(S-NS_0\right)\cdot\sum_{D_R=NS_0+1}^{\infty}\left(D_R-NS_0\right)f_R\left(D_R\right)}{\sum_{D_R=1}^{\infty}D_R\cdot f_R\left(D_R\right)} \quad (4)$$

where the denominator represents the expected total demand per replenishment cycle. Note that expression (4) can be used by any discrete demand distribution.

3.2 Derivation of an Exact Method to Compute *β*

As Section 1 points out, the fill rate is defined as the fraction of demand that is immediately fulfilled from shelf. From a practical point of view, it is useless to consider a service metric when there is no demand to be served. Therefore, cycles that do not show any demand should not be taken into account. According to [15], in order to derive an exact method to compute β over different demand patterns including intermittent demand is necessary to include explicitly the condition of having positive demand during the cycle. Then the fill rate can be expressed as

$$\beta=E\left(\frac{\text{fulfilled demand per replenishment cycle}}{\text{total demand per replenishment cycle}}\,|\,\text{positive demand during the cycle}\right) \quad (5)$$

Hence, positive demand during a cycle can be: (i) lower or equal than the net stock at the beginning of this cycle, i.e. $D_R \le NS_0$, and therefore the fill rate will be equal to 1; or (ii) greater than the net stock, i.e. $D_R > NS_0$, and therefore the fill rate will be the fraction of that demand which is satisfied by the on hand stock at the beginning of this cycle. Therefore

$$\beta\left(NS_0\right)=P\left(D_R\le NS_0\,\middle|\,D_R>0\right)+\sum_{D_R=NS_0+1}^{\infty}\frac{NS_0}{D_R}\cdot P\left(D_R\,\middle|\,D_R>0\right) \quad (6)$$

where the first term indicates the case (i) and the second term indicates the case (ii). Rewriting expression (6) through the probability mass and cumulative distribution functions of demand, $f_t(\cdot)$ and $F_t(\cdot)$, respectively, results into

$$\beta\left(NS_0\right)=\frac{F_R\left(NS_0\right)-F_R\left(0\right)}{1-F_R\left(0\right)}+\sum_{D_R=NS_0+1}^{\infty}\frac{NS_0}{D_R}\cdot\frac{f_R\left(D_R\right)}{1-F_R\left(0\right)} \quad (7)$$

Therefore, by applying expression (7) to every positive net stock level at the beginning of the cycle, the method to estimate β when demand follows any discrete distribution function results as follows:

$$\beta=\sum_{NS_0=1}^{S} f_L\left(S-NS_0\right)\cdot\left\{\frac{F_R\left(NS_0\right)-F_R\left(0\right)}{1-F_R\left(0\right)}+\sum_{D_R=NS_0+1}^{\infty}\frac{NS_0}{D_R}\cdot\frac{f_R\left(D_R\right)}{1-F_R\left(0\right)}\right\} \tag{8}$$

4 Illustrative Examples

This section illustrates the performance of expression (4) and (8) (β_{Trad} and β respectively) against the simulated fill rate, β_{Sim}, which is computed as the average fraction of the fulfilled demand in every replenishment cycle when considering 20,000 consecutive periods (T=20,000) as is expression (9). Data used for the simulation is presented in Table 1 which encompasses 180 different cases.

$$\beta_{Sim}=\frac{1}{T}\sum_{t=1}^{T}\frac{fulfilled\ demand_t}{total\ demand_t} \tag{9}$$

Table 1. Set of data (180 cases)

Lead time	L	=	1, 3, 5
Review period	R	=	1, 3, 5
Base stock	S	=	1, 3, 5, 7, 10
Demand Pattern negative binomial (r, θ)	smooth (4,0.7); intermittent (1.25,0.9); erratic (1.5, 0.3); lumpy (0.75,0.25)		

Demand is simulated by using the negative binomial since it is able to fulfil the smooth, intermittent, erratic and lumpy categories suggested by [16] as shown in Figure 2.

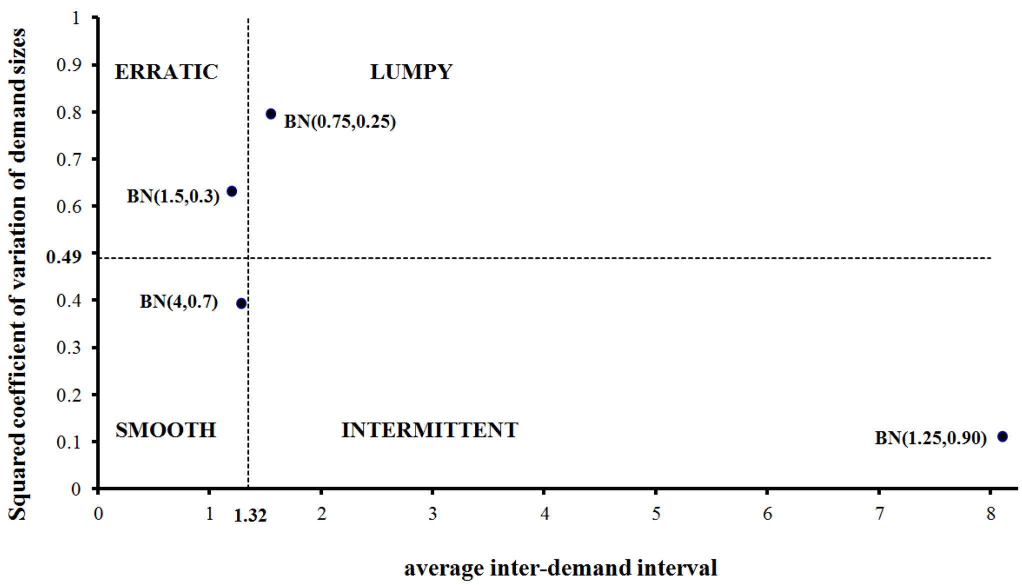

Fig. 2. Demand patterns used in the simulation according to the categorization framework of demand suggested by [16]

Figure 3 and Figure 4 show the comparison between β_{Trad} and β versus β_{Sim} respectively for the Table 1 cases. In Figure 3, we see that β_{Trad} tends to underestimates the simulated fill rate and therefore the traditional approximation seems to be biased. [9] pointed out similar results when demand is normally distributed whereas [15] when demand is Poisson distributed for the lost sales case. Note that expression (4) leads to the exact value of the traditional approximation. Therefore deviations that Figure 3 shows arise from estimating the fill rate using the traditional approach (expression (1)) and not from how it is calculated.

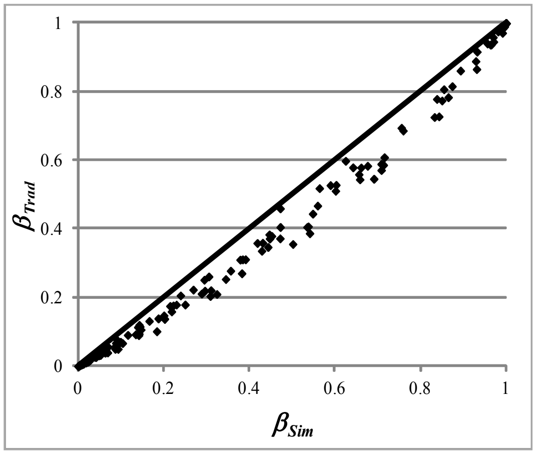

Fig. 3. β_{Trad} vs. β_{Sim} for the cases from Table 1

Regarding the performance of β, Figure 4 shows that neither bias nor significant deviations appears on it for any of the 180 cases and therefore β computes accurately the fill rate over different discrete demand patterns.

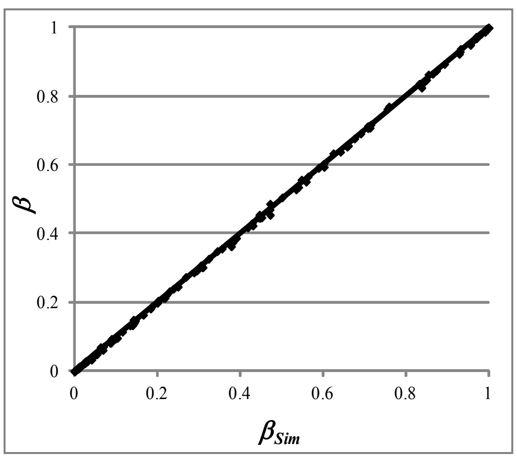

Fig. 4. β vs. β_{Sim} for the cases from Table 1

5 Discussion and summary

The traditional approach of the fill rate, β_{Trad}, computes it by estimating the ratio between the expected unfulfilled demand and the total expected demand per replenishment cycle through computing the expected shortage per replenishment cycle. Section 3.1 presents an exact method to compute β_{Trad} for any discrete demand distribution and for the backordering case. However Figure 3 shows that β_{Trad} tends to underestimate the simulated fill rate. An important consequence of the underestimation behavior is found when using a target fill rate to determine the base stock of the inventory policy. Figure 5 shows the evolution of β_{Trad}, β_{Sim}, and the exact estimation of the fill rate that is derived in Section 3.2, β, when increasing the base stock for a smooth demand modeled by a negative binomial with r=4 and θ=0.7. In this case if a target fill rate is set to 0.60, β_{Trad} leads to S=5 whereas in fact just S=3 is necessary to reach the target. In this example, using β_{Trad} to determine base stocks leads to an unnecessary increase in the average stock level and thus the holding costs of the system. This inefficiency is especially relevant in industries in which the unit cost of the item is high and/or storage space is limited. Therefore, managers should be aware of the risk of using the traditional approximation to set the base stock.

The method derived in Section 3.2, β, computes the fill rate directly as the expected fulfilled demand per replenishment cycle for the backordering case. As Figure 4 shows, this method presents the following advantages: (i) simulation results shows their accuracy over different demand patterns; (ii) outperforms the traditional approach and therefore avoid the above mentioned risks of using β_{Trad}; (iii) avoids the distortion caused in the metric by the cycles with no demand and therefore can be used even if the probability of no demand during the cycle cannot be neglected; (iv) applies for any discrete demand distribution.

Therefore, the exact fill rate method proposed in this paper leads to the exact fill rate value when demand follows any discrete distribution and can be applied even when the probability of zero demand cannot be neglected. Note that the need to consider only cycles with positive demand does not emerge when using the traditional approximation because it just considers the expected demand, and in the case of no demand cycles it does not affect the estimation.

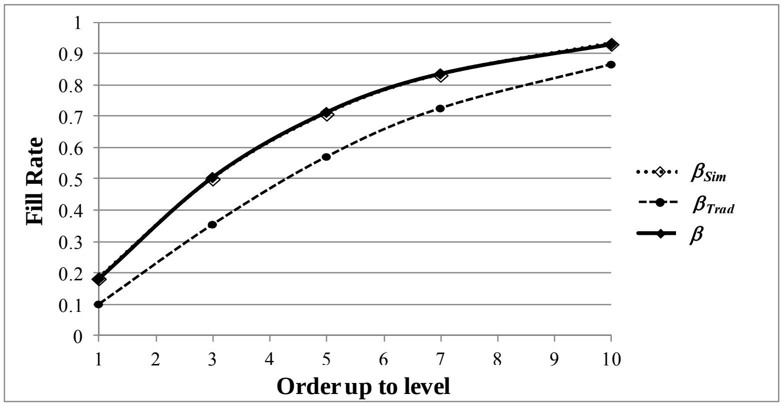

Fig. 5. Comparison between β_{Trad}, β and β_{Sim} with negative binomial demand with r=4 and θ=0.7 (smooth), R=1 and L=1

This paper is part of a wider research project devoted to identify the most simple and effective method to find the lowest base stock that guarantee the achievement of the target fill rate under any discrete demand context. Therefore, further extensions of this work should be focused on: (i) assessing the exact method when using other discrete distribution functions of demand; (ii) characterizing the cases where the approximation (including some possible new ones) has the most important deviations; (iii) analyzing risks of using different fill rate approximations to set the parameters of the stock policy and finally (iv) exploring the possibility of embedding results achieved in this work in information systems with the aim of helping decision processes.

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