Comparing Two Multivariable Complexity Functions Using One-variable Complexity Classes

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The comparison of algorithms complexities is very important both in theory and in practice. When we compare algorithms complexities we need to compare complexity functions. Usually we use one-variable complexity functions. Sometimes, we need multivariable complexity functions. In a previous paper we defined several one-variable complexity classes for multivariable complexity functions. Each complexity class of this type is a set of multivariable complexity functions, represented by a one-variable complexity function. In this paper we continue the work from that paper: we define new one-variable complexity classes and we prove several properties. The most important results are several criteria for two multivariable complexity functions to be comparable.
Keywords*: Algorithm, One-Variable Complexity Class, One-Variable Complexity Function, Multivariable Complexity Function, Functions Comparison*

1 Introduction

The complexity of an algorithm is usually expressed using complexity functions and complexity classes. A complexity function is a function defined on the set of positive integers and with values on the set of positive real numbers and it returns the quantity of computational resources necessary for an algorithm to solve a problem for a given dimension of the inputs. Using complexity functions give us an exact method to represent the complexity of an algorithm, but in many situations these functions have complicated expressions and it is difficult to work with them.

For this reason, computer scientists usually use complexity classes instead of complexity functions. Complexity classes are sets of complexity functions. They can be seen as equivalence classes with respect to the complexity functions. For each complexity class we have a representative complexity function. This function is chosen to be the function with the simplest mathematical expression.

Usually, we use only one-variable complexity functions and complexity classes, such as

$$\Theta(g(n)), O(g(n)), \Omega(g(n)), o(g(n)), \omega(g(n)) \quad (1)$$

The definitions for one-variable complexity classes can be found in almost any textbook related to the analysis of algorithms. See, for example, [1], [2], and [3].

When we compare algorithms we need to compare complexity functions. In [4] and [5] we obtained several results related to the comparison of two one-variable complexity functions using complexity classes. Our main results were several criteria for two one-variable complexity functions to be comparable.

Sometimes we need to work with multivariable complexity functions. In [6], we defined five one-variable complexity classes for multivariable complexity functions:

$$\overline{\Theta}(g(n)), \overline{O}(g(n)), \overline{\Omega}(g(n)), \overline{o}(g(n)), \overline{\omega}(g(n)) \quad (2)$$

The main idea was to use a one-variable complexity function as a representative function for each of these complexity classes, because there is easier to work with one-

variable functions. In addition, in [6] we gave some properties of these new defined classes. In this paper, we continue the work started in [6], proving for multivariable complexity functions and one-variable complexity classes several properties that we proved in [4] and [5] for one-variable complexity functions and one-variable complexity classes. This paper also contains the results from [6]: Definition 1, Definition 2, Proposition 2 a), b), Proposition 3, Theorem 2, and Theorem 3. We present here more detailed proofs for these two propositions and for Theorem 3.
The paper is organized as follows. In section 2, we remind the definitions for one-variable complexity classes for one-variable complexity functions, we present the definitions from [6], and we present some new definitions related to one-variable complexity classes for multivariable complexity functions. In Section 3 we prove some properties related to the complexity classes defined in Section 2. In Section 4, we present the main results of the paper. Section 5, contains the conclusion and some future work.

2 Definitions

We will denote by N_+ the set of positive integers and by R_+ the set of positive real numbers. Consider the function $g: N_+ \to R_+$ to be an arbitrary fixed complexity function.
Consider the following complexity classes (see [2], [3]):

$$\begin{aligned}\Theta(g(n)) = \{ f : N_+ \to R_+ \mid \exists c_1, c_2 \in R_+, \exists n_0 \in N_+ \\ such\ \ that\ \ c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n), \forall n \ge n_0 \}\end{aligned} \quad (3)$$

$$\begin{aligned}O(g(n)) = \{ f : N_+ \to R_+ \mid \exists c \in R_+, \exists n_0 \in N_+ \\ such\ \ that\ \ f(n) \le c \cdot g(n), \forall n \ge n_0 \}\end{aligned} \quad (4)$$

$$\begin{aligned}\Omega(g(n)) = \{ f : N_+ \to R_+ \mid \exists c \in R_+, \exists n_0 \in N_+ \\ such\ \ that\ \ c \cdot g(n) \le f(n), \forall n \ge n_0 \}\end{aligned} \quad (5)$$

$$\begin{aligned}o(g(n)) = \{ f : N_+ \to R_+ \mid \forall c \in R_+, \exists n_0 \in N_+ \\ such\ \ that\ \ f(n) < c \cdot g(n), \forall n \ge n_0 \}\end{aligned} \quad (6)$$

$$\begin{aligned}\omega(g(n)) = \{ f : N_+ \to R_+ \mid \forall c \in R_+, \exists n_0 \in N_+ \\ such\ \ that\ \ c \cdot g(n) < f(n), \forall n \ge n_0 \}\end{aligned} \quad (7)$$

Definition 1
Let be $k \in N_+$. Let be $x = (x_1, x_2, ..., x_k) \in (N_+)^k$ and $y = (y_1, y_2, ..., y_k) \in (N_+)^k$. We say that $x \ge y$ if $x_i \ge y_i, \forall i = 1..k$.

Remark 1
In the rest of the paper we will consider that the function $g(n)$ is monotonically increasing on N_+ (i.e. $\forall x, y \in N_+$ such that $x \le y$ we have $g(x) \le g(y)$).

Definition 2
Next, we define five one-variable complexity classes for multivariable complexity functions:

$$\begin{aligned}&\overline{\Theta}(g(n)) = \{ f : (N_+)^k \to R_+, k \in N_+ \mid \exists c_1, c_2 \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \\ &such\ \ that\ \ c_1 \cdot g(\min(n_1, n_2, ..., n_k)) \le f(n_1, n_2, ..., n_k) \le c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \quad (8) \\ &\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \}\end{aligned}$$

$$\overline{O}(g(n)) = \{ f : (N_+)^k \to R_+, k \in N_+ \mid \exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k$$
$$such\ \ that\ \ f(n_1, n_2, ..., n_k) \le c \cdot g(\max(n_1, n_2, ..., n_k)), \tag{9}$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \}$$

$$\overline{\Omega}(g(n)) = \{ f : (N_+)^k \to R_+, k \in N_+ \mid \exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k$$
$$such\ \ that\ \ c \cdot g(\min(n_1, n_2, ..., n_k)) \le f(n_1, n_2, ..., n_k), \tag{10}$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \}$$

$$\overline{o}(g(n)) = \{ f : (N_+)^k \to R_+, k \in N_+ \mid \forall c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k$$
$$such\ \ that\ \ f(n_1, n_2, ..., n_k) < c \cdot g(\min(n_1, n_2, ..., n_k)), \tag{11}$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \}$$

$$\overline{\varpi}(g(n)) = \{ f : (N_+)^k \to R_+, k \in N_+ \mid \forall c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k$$
$$such\ \ that\ \ c \cdot g(\max(n_1, n_2, ..., n_k)) < f(n_1, n_2, ..., n_k), \tag{12}$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \}$$

Definition 3
Let $f : (N_+)^k \to R_+$ be a complexity function. The function $f(n_1, n_2, ..., n_k)$ is comparable with the function $g(n)$ if

$$f(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n)) \cup \overline{O}(g(n)) \cup \overline{\Omega}(g(n)) \cup \overline{o}(g(n)) \cup \overline{\varpi}(g(n)) \tag{13}$$

We denote by $\overline{C}(g(n))$ the set of all complexity functions comparable with $g(n)$. Consequently,

$$\overline{C}(g(n)) = \overline{\Theta}(g(n)) \cup \overline{O}(g(n)) \cup \overline{\Omega}(g(n)) \cup \overline{o}(g(n)) \cup \overline{\varpi}(g(n)) \tag{14}$$

Definition 4
We define the following one-variable complexity classes:

$$\overline{o\Theta}(g(n)) = \overline{O}(g(n)) \setminus (\overline{o}(g(n)) \cup \overline{\Theta}(g(n))) \tag{15}$$
$$\overline{\Theta\varpi}(g(n)) = \overline{\Omega}(g(n)) \setminus (\overline{\Theta}(g(n)) \cup \overline{\varpi}(g(n))) \tag{16}$$

Definition 5
Let be $f_1 : (N_+)^k \to R_+$, $f_2 : (N_+)^k \to R_+$ two multivariable complexity functions. We say that the functions $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable if

$$\exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k\ \ such\ \ that$$
$$f_1(n_1, n_2, ..., n_k) \le c \cdot f_2(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \tag{17}$$

or

$$\begin{aligned}&\exists c \in R_+, \exists(n_1^0, n_2^0,..., n_k^0) \in (N_+)^k \ \textit{such that} \\ &f_2(n_1, n_2,..., n_k) \le c \cdot f_1(n_1, n_2,..., n_k), \forall (n_1, n_2,..., n_k) \ge (n_1^0, n_2^0,..., n_k^0)\end{aligned} \quad (18)$$

3 Some properties of the complexity classes

Proposition 1

a) $\Theta(g(n)) \subseteq \overline{\Theta}(g(n))$, $O(g(n)) \subseteq \overline{O}(g(n))$, $\Omega(g(n)) \subseteq \overline{\Omega}(g(n))$

b) $o(g(n)) \subseteq \overline{o}(g(n))$, $\omega(g(n)) \subseteq \overline{\omega}(g(n))$

Proof

For each complexity class for multivariable functions, if we consider only the functions for $k = 1$ then we obtain the corresponding class for one-variable functions.

Next, we show some details for obtaining the result for $\overline{\Theta}(g(n))$: in the definition of the class $\overline{\Theta}(g(n))$, we consider only the functions for $k = 1$:

$$\begin{aligned}&f : N_+ \to R_+ \mid \exists c_1, c_2 \in R_+, \exists n_1^0 \in N_+ \ \textit{such that} \\ &c_1 \cdot g(\min(n_1)) \le f(n_1) \le c_2 \cdot g(\max(n_1)), \forall n_1 \ge n_1^0\end{aligned} \quad (19)$$

Consequently, we have

$$\begin{aligned}&f : N_+ \to R_+ \mid \exists c_1, c_2 \in R_+, \exists n_1^0 \in N_+ \ \textit{such that} \\ &c_1 \cdot g(n_1) \le f(n_1) \le c_2 \cdot g(n_1), \forall n_1 \ge n_1^0\end{aligned} \quad (20)$$

which are exactly the functions from the class $\Theta(g(n))$. It follows that $\Theta(g(n)) \subseteq \overline{\Theta}(g(n))$.

For the other complexity classes the results can be obtained using the same idea. Consequently, we have the following results: $O(g(n)) \subseteq \overline{O}(g(n))$, $\Omega(g(n)) \subseteq \overline{\Omega}(g(n))$, $o(g(n)) \subseteq \overline{o}(g(n))$, $\omega(g(n)) \subseteq \overline{\omega}(g(n))$.

Proposition 2

a) $\overline{\Theta}(g(n)) \neq \varnothing$, $\overline{O}(g(n)) \neq \varnothing$, $\overline{\Omega}(g(n)) \neq \varnothing$

b) $\overline{o}(g(n)) \neq \varnothing$, $\overline{\omega}(g(n)) \neq \varnothing$

c) $\overline{o\Theta}(g(n)) \neq \varnothing$, $\overline{\Theta\omega}(g(n)) \neq \varnothing$

Proof

a), b) Using (3), (4), (5), (6), (7) one can prove that

$\Theta(g(n)) \neq \varnothing$, $O(g(n)) \neq \varnothing$, $\Omega(g(n)) \neq \varnothing$, $o(g(n)) \neq \varnothing$, $\omega(g(n)) \neq \varnothing$ (21)

For example, $g(n) \in \Theta(g(n))$, $g(n) \in O(g(n))$, $g(n) \in \Omega(g(n))$, $g(n)/n \in o(g(n))$, $n * g(n) \in \omega(g(n))$ (see [4]).

Next, using Proposition 1, it follows that $\overline{\Theta}(g(n)) \neq \varnothing$, $\overline{O}(g(n)) \neq \varnothing$, $\overline{\Omega}(g(n)) \neq \varnothing$, $\overline{o}(g(n)) \neq \varnothing$, $\overline{\omega}(g(n)) \neq \varnothing$.

c) Let be $f_1 : (N_+)^k \to R_+$, $f_2 : (N_+)^k \to R_+$ such that $f_1(n_1, n_2,..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2,..., n_k) \in \overline{\Theta}(g(n))$. Let be N_1 and N_2 two infinite subsets of $(N_+)^k$, such that N_1 and N_2 form a partition of $(N_+)^k$. Let be

$$f(n_1, n_2,..., n_k) = \begin{cases} f_1(n_1, n_2,..., n_k), (n_1, n_2,..., n_k) \in N_1 \\ f_2(n_1, n_2,..., n_k), (n_1, n_2,..., n_k) \in N_2 \end{cases} \quad (22)$$

Using (8), (9), and (11) one can prove that $f(n_1, n_2,..., n_k) \in \overline{O}(g(n))$,

$f(n_1, n_2, ..., n_k) \notin \overline{\Theta}(g(n))$, and $f(n_1, n_2, ..., n_k) \notin \overline{o}(g(n))$. Consequently, we have the following result: $f(n_1, n_2, ..., n_k) \in \overline{o\Theta}(g(n))$. It follows that $\overline{o\Theta}(g(n)) \neq \varnothing$.

For proving that $\overline{\Theta\omega}(g(n)) \neq \varnothing$ we can use the same idea used for $\overline{o\Theta}(g(n))$.

Proposition 3

a) $\overline{\Theta}(g(n)) \subseteq \overline{O}(g(n))$

b) $\overline{\Theta}(g(n)) \subseteq \overline{\Omega}(g(n))$

Proof

a) In [4] we proved that $g(n)/n \in o(g(n))$. Since $o(g(n)) \subseteq O(g(n))$ we have $g(n)/n \in O(g(n)) \subseteq \overline{O}(g(n))$. Next, we prove that $g(n)/n \notin \overline{\Theta}(g(n))$. Suppose that $g(n)/n \in \overline{\Theta}(g(n))$. Then, we have

$$\begin{aligned} &\exists c_1, c_2 \in R_+, \exists n_0 \in N_+ \; such \;\; that \\ &c_1 \cdot g(\min(n)) \leq g(n)/n \leq c_2 \cdot g(\max(n)), \forall n \geq n_0 \end{aligned} \quad (23)$$

i.e.

$$\exists c_1, c_2 \in R_+, \exists n_0 \in N_+ \; such \;\; that \;\; c_1 \cdot g(n) \leq g(n)/n \leq c_2 \cdot g(n), \forall n \geq n_0 \quad (24)$$

It follows that $c_1 \leq 1/n \leq c_2, \forall n \geq n_0$. Since $1/n \to 0$ for $n \to \infty$, we have $c_1 = 0$. That is a contradiction because $c_1 \in R_+$. Consequently, $g(n)/n \notin \overline{\Theta}(g(n))$.

Let be $f(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n))$. Then we have

$$\begin{aligned} &\exists c_1, c_2 \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \; such \;\; that \\ &c_1 \cdot g(\min(n_1, n_2, ..., n_k)) \leq f(n_1, n_2, ..., n_k) \leq c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \quad (25) \\ &\forall (n_1, n_2, ..., n_k) \geq (n_1^0, n_2^0, ..., n_k^0) \end{aligned}$$

so, we have:

$$\begin{aligned} &\exists c_2 \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \; such \;\; that \\ &f(n_1, n_2, ..., n_k) \leq c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \geq (n_1^0, n_2^0, ..., n_k^0) \end{aligned} \quad (26)$$

That means that $f(n_1, n_2, ..., n_k) \in \overline{O}(g(n))$. It follows that $\overline{\Theta}(g(n)) \subseteq \overline{O}(g(n))$.

b) The proof follows the same idea with the proof for a).

Proposition 4

a) $\overline{o}(g(n)) \subseteq \overline{O}(g(n))$

b) $\overline{\omega}(g(n)) \subseteq \overline{\Omega}(g(n))$

Proof

a) Note that $g(n) \in O(g(n)) \subseteq \overline{O}(g(n))$ and $g(n) \notin \overline{o}(g(n))$. It follows that $\overline{O}(g(n)) \setminus \overline{o}(g(n)) \neq \varnothing$.

Let be $f(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$. It follows that

$$\begin{aligned} &\forall c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \; such \;\; that \\ &f(n_1, n_2, ..., n_k) < c \cdot g(\min(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \geq (n_1^0, n_2^0, ..., n_k^0) \end{aligned} \quad (27)$$

Using (27) we have:

$$\exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \text{ such that}$$
$$f(n_1, n_2, ..., n_k) \le c \cdot g(\min(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (28)$$

Recall that $g(n)$ is monotonically increasing on N_+, so we have

$$g(\min(n_1, n_2, ..., n_k)) \le g(\max(n_1, n_2, ..., n_k)) \quad (29)$$

Consequently,

$$\exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \text{ such that}$$
$$f(n_1, n_2, ..., n_k) \le c \cdot g(\max(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (30)$$

It follows that $f(n_1, n_2, ..., n_k) \in \overline{O}(g(n))$. So, we have $\overline{o}(g(n)) \subseteq \overline{O}(g(n))$.

b) The proof follows the same idea with the proof for a).

Proposition 5

a) $\overline{o}(g(n)) \cap \overline{\Theta}(g(n)) = \emptyset$

b) $\overline{\Theta}(g(n)) \cap \overline{\omega}(g(n)) = \emptyset$

c) $\overline{O}(g(n)) \cap \overline{\Omega}(g(n)) = \overline{\Theta}(g(n))$

Proof

a) Suppose that there exists a multivariable complexity function $f(n_1, n_2, ..., n_k)$, such that $f(n_1, n_2, ..., n_k) \in \overline{o}(g(n)) \cap \overline{\Theta}(g(n))$. It follows that

$$\forall c \in R_+, \exists (n_1^{01}, n_2^{01}, ..., n_k^{01}) \in (N_+)^k \text{ such that}$$
$$f(n_1, n_2, ..., n_k) < c \cdot g(\min(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^{01}, n_2^{01}, ..., n_k^{01}) \quad (31)$$

and

$$\exists c_1, c_2 \in R_+, \exists (n_1^{02}, n_2^{02}, ..., n_k^{02}) \in (N_+)^k \text{ such that}$$
$$c_1 \cdot g(\min(n_1, n_2, ..., n_k)) \le f(n_1, n_2, ..., n_k) \le c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \quad (32)$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^{02}, n_2^{02}, ..., n_k^{02})$$

From (31) it follows that for $c = c_1$ we have

$$\exists (n_1^{01}, n_2^{01}, ..., n_k^{01}) \in (N_+)^k \text{ such that}$$
$$f(n_1, n_2, ..., n_k) < c_1 \cdot g(\min(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^{01}, n_2^{01}, ..., n_k^{01}) \quad (33)$$

Consider $(n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k$ such that

$$(n_1^0, n_2^0, ..., n_k^0) = (\max(n_1^{01}, n_1^{02}), \max(n_2^{01}, n_2^{02}), ..., \max(n_k^{01}, n_k^{02})) \quad (34)$$

Using (32) and (33) we have:

$$c_1 \cdot g(\min(n_1, n_2, ..., n_k)) \le f(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (35)$$

and

$$f(n_1,n_2,...,n_k) < c_1 \cdot g(\min(n_1,n_2,...,n_k)), \forall (n_1,n_2,...,n_k) \geq (n_1^0,n_2^0,...,n_k^0) \quad (36)$$

So, we have obtained a contradiction. Consequently, we have $\overline{o}(g(n)) \cap \overline{\Theta}(g(n)) = \emptyset$.
b) The proof follows the same idea with the proof for a).
c) From Proposition 3, we have $\overline{\Theta}(g(n)) \subseteq \overline{O}(g(n))$ and $\overline{\Theta}(g(n)) \subseteq \overline{\Omega}(g(n))$. It follows that
$\overline{\Theta}(g(n)) \subseteq \overline{O}(g(n)) \cap \overline{\Omega}(g(n))$ (37)

Next, we prove that $\overline{O}(g(n)) \cap \overline{\Omega}(g(n)) \subseteq \overline{\Theta}(g(n))$. Let be $f(n_1,n_2,...,n_k)$ a multivariable complexity function such that $f(n_1,n_2,...,n_k) \in \overline{O}(g(n)) \cap \overline{\Omega}(g(n))$. It follows that

$$\begin{aligned}&\exists c_2 \in R_+, \exists (n_1^{02},n_2^{02},...,n_k^{02}) \in (N_+)^k\\ &\textit{such that } f(n_1,n_2,...,n_k) \leq c_2 \cdot g(\max(n_1,n_2,...,n_k)), \quad (38)\\ &\forall (n_1,n_2,...,n_k) \geq (n_1^{02},n_2^{02},...,n_k^{02})\end{aligned}$$

and

$$\begin{aligned}&\exists c_1 \in R_+, \exists (n_1^{01},n_2^{01},...,n_k^{01}) \in (N_+)^k\\ &\textit{such that } c_1 \cdot g(\min(n_1,n_2,...,n_k)) \leq f(n_1,n_2,...,n_k), \quad (39)\\ &\forall (n_1,n_2,...,n_k) \geq (n_1^{01},n_2^{01},...,n_k^{01})\end{aligned}$$

Using the same idea that we used in (34), consider $(n_1^0,n_2^0,...,n_k^0) \in (N_+)^k$ such that $(n_1^0,n_2^0,...,n_k^0) = (\max(n_1^{01},n_1^{02}), \max(n_2^{01},n_2^{02}),..., \max(n_k^{01},n_k^{02}))$. Recall that the function $g(n)$ is monotonically increasing on N_+, so we have $g(\min(n_1,n_2,...,n_k)) \leq g(\max(n_1,n_2,...,n_k))$. Next, using (38) and (39), we have

$$\begin{aligned}&\exists c_1, c_2 \in R_+, \exists (n_1^0,n_2^0,...,n_k^0) \in (N_+)^k \textit{ such that}\\ &c_1 \cdot g(\min(n_1,n_2,...,n_k)) \leq f(n_1,n_2,...,n_k) \leq c_2 \cdot g(\max(n_1,n_2,...,n_k)), \quad (40)\\ &\forall (n_1,n_2,...,n_k) \geq (n_1^0,n_2^0,...,n_k^0)\end{aligned}$$

It follows that $f(n_1,n_2,...,n_k) \in \Theta(g(n))$. Consequently,
$\overline{O}(g(n)) \cap \overline{\Omega}(g(n)) \subseteq \overline{\Theta}(g(n))$.
Next, using (37), we have $\overline{O}(g(n)) \cap \overline{\Omega}(g(n)) = \overline{\Theta}(g(n))$.

Proposition 6

The complexity classes $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, and $\overline{\Theta}(g(n))$ form a partition of the complexity class $\overline{O}(g(n))$.

Proof

From Proposition 5 we have $\overline{o}(g(n)) \cap \overline{\Theta}(g(n)) = \emptyset$. From (15) we have $\overline{o\Theta}(g(n)) = \overline{O}(g(n)) \setminus (\overline{o}(g(n)) \cup \overline{\Theta}(g(n)))$.
Consequently, $\overline{o}(g(n)) \cap \overline{o\Theta}(g(n)) = \emptyset$ and $\overline{o\Theta}(g(n)) \cap \overline{\Theta}(g(n))) = \emptyset$. It follows that the complexity classes $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, and $\overline{\Theta}(g(n))$ are pairwise disjoint.
From Proposition 3 we have $\overline{\Theta}(g(n)) \subseteq \overline{O}(g(n))$; from Proposition 4 we have $\overline{o}(g(n)) \subseteq \overline{O}(g(n))$. It follows that $(\overline{o}(g(n)) \cup \overline{\Theta}(g(n))) \subseteq \overline{O}(g(n))$. Next, using (15), we have $\overline{O}(g(n)) = \overline{o}(g(n)) \cup \overline{o\Theta}(g(n)) \cup \overline{\Theta}(g(n))$.

Proposition 7

The complexity classes $\overline{\Theta}(g(n))$, $\overline{\Theta\omega}(g(n))$, and $\overline{\varpi}(g(n))$ form a partition of the complexity class $\overline{\Omega}(g(n))$.

Proof

The proof follows the same idea with the proof for Proposition 6.

Remark 2

Using Proposition 6, Proposition 7, and Definition 3, we have:

$$\overline{C}(g(n)) = \overline{O}(g(n)) \cup \overline{\Omega}(g(n)) \quad (41)$$

4. The main results

Theorem 1

The complexity classes $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, $\overline{\Theta}(g(n))$, $\overline{\Theta\omega}(g(n))$, and $\overline{\varpi}(g(n))$ form a partition of the set $\overline{C}(g(n))$.

Proof

From Proposition 6, we have that $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, and $\overline{\Theta}(g(n))$ are pairwise disjoint. From Proposition 7, we have that $\overline{\Theta}(g(n))$, $\overline{\Theta\omega}(g(n))$ and $\overline{\varpi}(g(n))$ are pairwise disjoint. From Proposition 5, we have that $\overline{O}(g(n)) \cap \overline{\Omega}(g(n)) = \overline{\Theta}(g(n))$. Consequently, we have $\overline{o}(g(n)) \cap \overline{\Theta\omega}(g(n)) = \varnothing$, $\overline{o}(g(n)) \cap \overline{\varpi}(g(n)) = \varnothing$, $\overline{o\Theta}(g(n)) \cap \overline{\Theta\omega}(g(n)) = \varnothing$, and $\overline{o\Theta}(g(n)) \cap \overline{\varpi}(g(n)) = \varnothing$. It follows that $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, $\overline{\Theta}(g(n))$, $\overline{\Theta\omega}(g(n))$, and $\overline{\varpi}(g(n))$ are pair wise disjoint.

From Proposition 6, Proposition 7, and (14) we have

$$\overline{C}(g(n)) = \overline{o}(g(n)) \cup \overline{o\Theta}(g(n)) \cup \overline{\Theta}(g(n)) \cup \overline{\Theta\omega}(g(n)) \cup \overline{\varpi}(g(n)) \quad (42)$$

It follows that $\overline{o}(g(n))$, $\overline{o\Theta}(g(n))$, $\overline{\Theta}(g(n))$, $\overline{\Theta\omega}(g(n))$, and $\overline{\varpi}(g(n))$ form a partition of the set $\overline{C}(g(n))$.

Theorem 2

Let be $f:(N_+)^k \to R_+$ a complexity function with the following property:

$$\begin{aligned} &f(m,m,...,m) \le f(n_1,n_2,...,n_k) \le f(M,M,...,M), \forall (n_1,n_2,...,n_k) \in (N_+)^k, \\ &\textit{where } m = \min(n_1,n_2,...,n_k) \textit{ and } M = \max(n_1,n_2,...,n_k) \end{aligned} \quad (43)$$

Then we have

$$\begin{aligned} &f(n_1,n_2,...,n_k) \in \overline{\Theta}(g(n)),\ f(n_1,n_2,...,n_k) \in \overline{O}(g(n)),\ f(n_1,n_2,...,n_k) \in \overline{\Omega}(g(n)) \\ &\textit{where } g(n) = f(n,n,...,n) \end{aligned} \quad (44)$$

Proof

Consider the function $f(n_1,n_2,...,n_k)$ with the properties from the hypothesis. It follows that

$$\begin{aligned} &g(\min(n_1,n_2,...,n_k)) \le f(n_1,n_2,...,n_k) \le g(\max(n_1,n_2,...,n_k)), \\ &\forall (n_1,n_2,...,n_k) \in (N_+)^k \end{aligned} \quad (45)$$

Consequently,

$$\begin{aligned} &\exists c_1 = 1, c_2 = 1 \in R_+, \exists (n_1^0,n_2^0,...,n_k^0) = (1,1,...,1) \in (N_+)^k \textit{ such that} \\ &c_1 \cdot g(\min(n_1,n_2,...,n_k)) \le f(n_1,n_2,...,n_k) \le c_2 \cdot g(\max(n_1,n_2,...,n_k)), \quad (46) \\ &\forall (n_1,n_2,...,n_k) \ge (n_1^0,n_2^0,...,n_k^0) \end{aligned}$$

This means that $f(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n))$. Using this result and Proposition 3, we have $f(n_1, n_2, ..., n_k) \in \overline{O}(g(n))$ and $f(n_1, n_2, ..., n_k) \in \overline{\Omega}(g(n))$.

Theorem 3

a) Let $f:(N_+)^k \to R_+$ be a complexity function with the following property:

$$f(n_1, n_2, ..., n_k) < m \cdot f(m, m, ..., m), \forall (n_1, n_2, ..., n_k) \in (N_+)^k, \quad where\ m = \min(n_1, n_2, ..., n_k) \quad (47)$$

Then we have

$$f(n_1, n_2, ..., n_k) \in \overline{o}(g(n)), \quad where\ g(n) = n^2 \cdot f(n, n, ..., n) \quad (48)$$

In addition, we have $f(n_1, n_2, ..., n_k) \in \overline{O}(g(n))$.

b) Let $f:(N_+)^k \to R_+$ be a complexity function with the following property:

$$f(M, M, ..., M)/M < f(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \in (N_+)^k, \quad where\ M = \max(n_1, n_2, ..., n_k) \quad (49)$$

Then we have

$$f(n_1, n_2, ..., n_k) \in \overline{\omega}(g(n)), \quad where\ g(n) = f(n, n, ..., n)/n^2 \quad (50)$$

In addition, we have $f(n_1, n_2, ..., n_k) \in \overline{\Omega}(g(n))$.

Proof

a) Consider the function $f(n_1, n_2, ..., n_k)$ with the properties from the hypothesis. It follows that

$$f(n_1, n_2, ..., n_k) \le m \cdot f(m, m, ..., m), \forall (n_1, n_2, ..., n_k) \in (N_+)^k, \quad where\ m = \min(n_1, n_2, ..., n_k) \quad (51)$$

Consider the following inequality:

$$f(n_1, n_2, ..., n_k) \le (c \cdot m) \cdot m \cdot f(m, m, ..., m),\ where\ m = \min(n_1, n_2, ..., n_k) \quad (52)$$

In order for this inequality to be true for all $c \in R_+$, we search for each $c \in R_+$ a value for $m, m \in N_+$. The condition that must be true is $(c \cdot m) \ge 1$. So, we take $m \ge 1/c$. Consequently,

$$\begin{aligned} &\forall c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) = (\lfloor 1/c \rfloor + 1, \lfloor 1/c \rfloor + 1, ..., \lfloor 1/c \rfloor + 1) \in (N_+)^k \ such\ \ that \\ &f(n_1, n_2, ..., n_k) < c \cdot m^2 \cdot f(m, m, ..., m)), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0), \\ &where\ m = \min(n_1, n_2, ..., n_k) \end{aligned} \quad (53)$$

It follows that

$$\begin{aligned} &\forall c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) = (\lfloor 1/c \rfloor + 1, \lfloor 1/c \rfloor + 1, ..., \lfloor 1/c \rfloor + 1) \in (N_+)^k \ such\ \ that \\ &f(n_1, n_2, ..., n_k) < c \cdot g(\min(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \end{aligned} \quad (54)$$

From this expression, we have $f(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$, and using Proposition 4, we have $f(n_1, n_2, ..., n_k) \in \overline{O}(g(n))$.

b) The proof follows the same idea with the proof for a).

Theorem 4

a) Let be $f_1(n_1,n_2,...,n_k)\in\overline{o}(g(n))$ and $f_2(n_1,n_2,...,n_k)\in\overline{\Omega}(g(n))$. Then $f_1(n_1,n_2,...,n_k)$ and $f_2(n_1,n_2,...,n_k)$ are comparable.

b) Let be $f_1(n_1,n_2,...,n_k)\in\overline{\varpi}(g(n))$ and $f_2(n_1,n_2,...,n_k)\in\overline{O}(g(n))$. Then $f_1(n_1,n_2,...,n_k)$ and $f_2(n_1,n_2,...,n_k)$ are comparable.

Proof

a) From $f_1(n_1,n_2,...,n_k)\in\overline{o}(g(n))$ and $f_2(n_1,n_2,...,n_k)\in\overline{\Omega}(g(n))$ we have

$$\forall c_1\in R_+,\exists(n_1^{01},n_2^{01},...,n_k^{01})\in(N_+)^k \textit{ such that } f_1(n_1,n_2,...,n_k)<c_1\cdot g(\min(n_1,n_2,...,n_k)), \forall(n_1,n_2,...,n_k)\geq(n_1^{01},n_2^{01},...,n_k^{01})$$
(55)

and

$$\exists c_2\in R_+,\exists(n_1^{02},n_2^{02},...,n_k^{02})\in(N_+)^k \textit{ such that } c_2\cdot g(\min(n_1,n_2,...,n_k))\leq f_2(n_1,n_2,...,n_k), \forall(n_1,n_2,...,n_k)\geq(n_1^{02},n_2^{02},...,n_k^{02})$$
(56)

From (55) we have

$$\exists c_1\in R_+,\exists(n_1^{01},n_2^{01},...,n_k^{01})\in(N_+)^k \textit{ such that } f_1(n_1,n_2,...,n_k)<c_1\cdot g(\min(n_1,n_2,...,n_k)), \forall(n_1,n_2,...,n_k)\geq(n_1^{01},n_2^{01},...,n_k^{01})$$
(57)

Consider $(n_1^0,n_2^0,...,n_k^0)=(\max(n_1^{01},n_1^{02}),\max(n_2^{01},n_2^{02}),...,\max(n_k^{01},n_k^{02}))$. It follows that

$$f_1(n_1,n_2,...,n_k)<c_1\cdot g(\min(n_1,n_2,...,n_k)),\forall(n_1,n_2,...,n_k)\geq(n_1^0,n_2^0,...,n_k^0) \quad (58)$$

and

$$c_2\cdot g(\min(n_1,n_2,...,n_k))\leq f_2(n_1,n_2,...,n_k),\forall(n_1,n_2,...,n_k)\geq(n_1^0,n_2^0,...,n_k^0) \quad (59)$$

Let be $c\in R_+$ such that $c_1=c\cdot c_2$. Consequently, we have

$$\begin{aligned}&f_1(n_1,n_2,...,n_k)<c_1\cdot g(\min(n_1,n_2,...,n_k))=\\&=c\cdot c_2\cdot g(\min(n_1,n_2,...,n_k))\leq c\cdot f_2(n_1,n_2,...,n_k), \quad (60)\\&\forall(n_1,n_2,...,n_k)\geq(n_1^0,n_2^0,...,n_k^0)\end{aligned}$$

It follows that

$$\exists c\in R_+,\exists(n_1^0,n_2^0,...,n_k^0)\in(N_+)^k \textit{ such that } f_1(n_1,n_2,...,n_k)<c\cdot f_2(n_1,n_2,...,n_k),\forall(n_1,n_2,...,n_k)\geq(n_1^0,n_2^0,...,n_k^0) \quad (61)$$

Consequently, $f_1(n_1,n_2,...,n_k)$ and $f_2(n_1,n_2,...,n_k)$ are comparable.

b) From $f_1(n_1,n_2,...,n_k)\in\overline{\varpi}(g(n))$ and $f_2(n_1,n_2,...,n_k)\in\overline{O}(g(n))$ we have

$$\forall c_1\in R_+,\exists(n_1^{01},n_2^{01},...,n_k^{01})\in(N_+)^k \textit{ such that } c_1\cdot g(\max(n_1,n_2,...,n_k))<f_1(n_1,n_2,...,n_k), \forall(n_1,n_2,...,n_k)\geq(n_1^{01},n_2^{01},...,n_k^{01})$$
(62)

and

$$\exists c_2 \in R_+, \exists (n_1^{02}, n_2^{02}, ..., n_k^{02}) \in (N_+)^k \text{ such that } f_2(n_1, n_2, ..., n_k) \le c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^{02}, n_2^{02}, ..., n_k^{02})$$
(63)

From (62) we have

$$\exists c_1 \in R_+, \exists (n_1^{01}, n_2^{01}, ..., n_k^{01}) \in (N_+)^k \text{ such that } c_1 \cdot g(\max(n_1, n_2, ..., n_k)) < f_1(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \ge (n_1^{01}, n_2^{01}, ..., n_k^{01})$$
(64)

Consider

$$(n_1^0, n_2^0, ..., n_k^0) = (\max(n_1^{01}, n_1^{02}), \max(n_2^{01}, n_2^{02}), ..., \max(n_k^{01}, n_k^{02}))$$

It follows that

$$c_1 \cdot g(\max(n_1, n_2, ..., n_k)) < f_1(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (65)$$

and

$$f_2(n_1, n_2, ..., n_k) \le c_2 \cdot g(\max(n_1, n_2, ..., n_k)), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (66)$$

Let be $c \in R_+$ such that $c_2 = c \cdot c_1$. Consequently, we have

$$f_2(n_1, n_2, ..., n_k) \le c_2 \cdot g(\max(n_1, n_2, ..., n_k)) = = c \cdot c_1 \cdot g(\max(n_1, n_2, ..., n_k)) < c \cdot f_1(n_1, n_2, ..., n_k), \quad (67)$$
$$\forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0)$$

It follows that

$$\exists c \in R_+, \exists (n_1^0, n_2^0, ..., n_k^0) \in (N_+)^k \text{ such that } f_2(n_1, n_2, ..., n_k) < c \cdot f_1(n_1, n_2, ..., n_k), \forall (n_1, n_2, ..., n_k) \ge (n_1^0, n_2^0, ..., n_k^0) \quad (68)$$

Consequently, $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Theorem 5
Let be $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\omega}(g(n))$. Then $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Proof
From Proposition 4, we have $\overline{\omega}(g(n)) \subseteq \overline{\Omega}(g(n))$. From the hypothesis we have $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\omega}(g(n))$. It follows that $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\Omega}(g(n))$. Next, using Theorem 4, we have that $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Theorem 6
a) Let be $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n))$. Then $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.
b) Let be $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\Theta\omega}(g(n))$. Then $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Proof
a) From Proposition 3, we have $\overline{\Theta}(g(n)) \subseteq \overline{\Omega}(g(n))$. From the hypothesis we have $f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n))$. It follows that

$f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and
$f_2(n_1, n_2, ..., n_k) \in \overline{\Omega}(g(n))$. Next, using Theorem 4, we have that $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

b) From (16), we have $\overline{\Theta\omega}(g(n)) \subseteq \overline{\Omega}(g(n))$.
From the hypothesis we have
$f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and
$f_2(n_1, n_2, ..., n_k) \in \overline{\Theta\omega}(g(n))$. It follows that
$f_1(n_1, n_2, ..., n_k) \in \overline{o}(g(n))$ and
$f_2(n_1, n_2, ..., n_k) \in \overline{\Omega}(g(n))$. Next, using Theorem 4, we have that $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Theorem 7
a) Let be $f_1(n_1, n_2, ..., n_k) \in \overline{\omega}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{\Theta}(g(n))$. Then $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.
b) Let be $f_1(n_1, n_2, ..., n_k) \in \overline{\omega}(g(n))$ and $f_2(n_1, n_2, ..., n_k) \in \overline{o\Theta}(g(n))$ Then $f_1(n_1, n_2, ..., n_k)$ and $f_2(n_1, n_2, ..., n_k)$ are comparable.

Proof
The proof follows the same idea with the proof for Theorem 6.

5 Conclusion
In [6] we defined five one-variable complexity classes for multivariable complexity functions. In this paper we continue that work, defining new one-variable complexity classes and proving new properties. The most important results are several criteria for two multivariable functions to be comparable. The results presented in this paper are important because they reduce the work with multivariable complexity functions to the work with one-variable complexity classes and one-variable complexity functions.
As a future work, we want to obtain more powerful results related to the link between one-variable complexity classes and multivariable complexity functions. Another future work can be to study the behavior of multivariable complexity functions for various types of algorithms, in order to find new characterizations for these functions.

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