Multi-Level Model

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Is an original paper, which contains a hierarchical model with three levels, for determining the linearized non-homogeneous and homogeneous credibility premiums at company level, at sector level and at contract level, founded on the relevant covariance relations between the risk premium, the observations and the weighted averages. We give a rather explicit description of the input data for the multi- level hierarchical model used, only to show that in practical situations, there will always be enough data to apply credibility theory to a real insurance portfolio.
Mathematics Subject Classification: 62P05.
Keywords: *hierarchical structure with three levels, observable variables with associated weights, the credibility results.*

Introduction

In this article we first give the two-level hierarchical model of Jewell, involving a portfolio of contracts, which can be broken up into sub-portfolios (sectors), each consisting of groups. In **Section 1** we will give the assumptions of the hierarchical model with two levels and the question to be solved is: find (credibility) estimates for the pure risk premium of the class (which is a set of the contracts, often referred to as a contract again), for the pure risk premium of the sector.

Some unbiased estimators are given in **Section 2**. This completes the solution of the hierarchical credibility model in case of non-homogeneous linear credibility estimates. When one considers homogeneous linear credibility approximations, see **Section 3**, again one obtains the results with the parameters estimated as in the previous section. Jewell's hierarchical model is a two level classification procedure. It is clear that this process can be generalized to any number of levels. **Section 4**, contains a description of the hierarchical model with three levels. So, one might create a multi-level hierarchical model by, e.g. grouping sectors into cohorts, and have different structure parameters for each level. The risk parameters pertaining to a certain contract are a random vector, of which the last component is unique for the contract at hand the next-to-last for the sector the contract is in, and so on.

1. The description of the hierarchical model with two levels

We consider now a portfolio of contracts, which can be broken up to into sub-portfolios (sector), each consisting of groups. The sector is characterized by a risk parameter drawn from a structure distribution describing the heterogeneity between sectors. Given the sector, the group (of contracts) is characterized by another risk parameter. We get the scheme of Diagram 1.

Each contract $j = \overline{1, k_p}$ (each class j in sector p) is the average of a group of w_{pjr} contracts, where w_{pjr} is the weight (size) of the group j at time r, with $r = \overline{1,t}$.

The model consists of the structural variables θ_p and θ_{pj} and the observable variables X_{pjr}, where $p = \overline{1,P}, j = \overline{1,k_p}, r = \overline{1,t}$.

So the sector p consists of the set of variables:

$$\left(\theta_p, \underline{\theta}_p, X_p\right) = \theta_p, \theta_{pj}, X_{pjr}, j = \overline{1,k_p}, r = \overline{1,t}$$

and the contract (p, j) consists of the variables: $\left(\theta_{pj}, \underline{X}_{pj}\right) = \theta_{pj}, X_{pjr}, r = \overline{1,t}$.

Sector	1……………………p…………………P
Contracts	(p, 1)……………..(p, j)………..(p, k_p)
Structure variables	θ_p
	θ_{p1},…………….., θ_{pj},…………, θ_{pk_p}
Observable variables with associated weights	$\left.\begin{matrix} X_{pjr} \\ (w_{pjr}) \end{matrix}\right\\}$ $\begin{matrix} j=\overline{1,k_p} \\ r=\overline{1,t} \end{matrix}$

Diagram 1. Hierarchical scheme

Of course the variables X_{pjr} represents the average of w_{pjr} contracts grouped together at time r as follows:
$X_{pjr} = \frac{1}{w_{pjr}} \sum_{i=1}^{w_{pjr}} X_{pjr}^{(i)}, r = \overline{1,t}, j = \overline{1,k_p}, p = \overline{1,P}$.
The hypotheses of the Jewell model can be formulated as:
(J_1) The sectors are independent:
$\left(\theta_p, \underline{\theta}_p, X_p\right)$ is independent of $\left(\theta_{p'}, \underline{\theta}_{p'}, X_{p'}\right)$, with $p, p' = \overline{1,P}$ and $p \neq p'$;
(J_2) For each $p = \overline{1,P}$ and for given values of θ_p, the contracts $\left(\theta_{pj}, \underline{X}_{pj}\right)$ are conditionally independent;
(J_3) For each $p = \overline{1,P}$, $j = \overline{1,k_p}$ and for given values of $\left(\theta_p, \theta_{pj}\right)$, the observations $\underline{X}_{pj}$ are conditionally independent;
(J_4) All pairs of variables $\left(\theta_p, \theta_{pj}\right)$ for $p = \overline{1,P}, j = \overline{1,k_p}$ are identically distributed;
(J_5) $E\left(X_{pjr} \mid \theta_p, \theta_{pj}\right) = \mu\left(\theta_p, \theta_{pj}\right)$, for all $r = \overline{1,t}$ [$\mu\left(\theta_p, \theta_{pj}\right)$ is the pure net risk premium of the contract (p, j)]
$Var\left(X_{pjr} \mid \theta_p, \theta_{pj}\right) = \sigma^2\left(\theta_p, \theta_{pj}\right) / w_{pjr}$, for all $r = \overline{1,t}$;
(J_6) $E\left(X_{pjr} \mid \theta_p\right) = \nu\left(\theta_p\right), j = \overline{1,k_p}, r = \overline{1,t}$
[$\nu\left(\theta_p\right)$ is the pure net risk premium of sector p], with $X_{pjr}^{(i)}, i = \overline{1,w_{pjr}}, r = \overline{1,t}, j = \overline{1,k_p}, p = \overline{1,P}$ satisfying the hypotheses: (J_1), (J_2), (J_3), (J_4), (J_5') and (J_6'), where:
(J_5') All contracts have in common that their variances and expectations are represented by the same functions $\sigma^2(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ of the risk parameter ($\sigma^2(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ do not depend on the subscripts: p, j and r), that is:
$Var\left(X_{pjr}^{(i)} \mid \theta_p, \theta_{pj}\right) = \sigma^2\left(\theta_p, \theta_{pj}\right)$,
$i = \overline{1,w_{pjr}}, r = \overline{1,t}$
$E\left(X_{pjr}^{(i)} \mid \theta_p, \theta_{pj}\right) = \mu\left(\theta_p, \theta_{pj}\right), i = \overline{1,w_{pjr}}, r = \overline{1,t}$
(J_6') All sectors have in common that their expectations are represented by the same function $\nu(\cdot)$ of the risk parameter (the functions $\nu(\cdot)$ do not depend on the subscripts: p, j and r), that is:
$E\left(X_{pjr}^{(i)} \mid \theta_p\right) = \nu\left(\theta_p\right), i = \overline{1,w_{pjr}}, r = \overline{1,t}$.
This section provides us with estimates for $\nu\left(\theta_p\right)$ on sector level, and for $\mu\left(\theta_p, \theta_{pj}\right)$ on contract level. The structural parameters that will occur in the credibility premium and their interpretation now are as follows:
i)
$m = m_p = E\left[\nu\left(\theta_p\right)\right] = E\left[\mu\left(\theta_p, \theta_{pj}\right)\right] = E\left(X_{pjr}\right)$.
This represents the combined expectation for the entire collective;
ii) $s^2 = E\left[\sigma^2\left(\theta_p, \theta_{pj}\right)\right]$. This structure parameter s^2 measures the degree of fluctuation of the individual contract or the heterogeneity in time of the data;

iii) $a = E\left[Var\left(\mu\left(\theta_p, \theta_{pj}\right) \mid \theta_p\right)\right]$. This quantity a now measures the degree of variability in a sector, or the heterogeneity within a sector;
iv) $b = Var\left[\nu\left(\theta_p\right)\right]$. This structure parameter b is a measure for the heterogeneity between the different sectors.
Define z_{pj} which will later prove to be a credibility factor on contract level and z_p the credibility factor at sector level, as:
$z_{pj} = aw_{pj.} / \left(s^2 + aw_{pj.}\right), z_p = bz_{p.} / \left(a + bz_{p.}\right)$
The weights appearing in the definition of z_{pj} are the natural weights w_{pjr} $\left(w_{pj.} = \sum_{r=1}^{t} w_{pjr}\right)$.Those for z_p are the cumulated credibility weights. It is important to keep in mind the distinction between $z_{p.} = \sum_j z_{pj}$ and z_p. Further introduce the following weighted averages:

$$X_{pjw} = \sum_{r=1}^{t} \frac{w_{pjr}}{w_{pj.}} X_{pjr}$$

The averages we will use for the sector and the entire collective are again weighted with the cumulated credibility factors instead of the natural weights:

$$X_{pzw} = \sum_{j=1}^{k} \frac{z_{pj}}{z_{p.}} X_{pjw}, X_{zzw} = \sum_{p=1}^{P} \frac{z_p}{z_.} X_{pzw} \left(z_. = \sum_{1}^{P} z_p\right)$$

The following estimators will be used in the sequel:
$N_p = X_{pzw}$ individual estimator for $\nu\left(\theta_p\right)$;
$M_{pj} = X_{pjw}$ individual estimator for $\mu\left(\theta_p, \theta_{pj}\right)$;
$N_0 = X_{zzw}$ collective estimator for $\nu\left(\theta_p\right)$ş
$M_{p0} = X_{pzw}$ collective estimator for $\mu\left(\theta_p, \theta_{pj}\right)$;
Note that $N_p = M_{p0}$. Now, we derive the credibility results for the two-level hierarchical model. The credibility premiums at sector level are given in the following theorem:
Theorem 1.1: (Credibility estimate at sector level) *Consider the two-level hierarchical model of Jewell as introduced in this section. Under the hypotheses (J_1)-(J_6), the following linearized non-homogeneous estimator is obtained for the pure net risk premium* of *sector p:* $\hat{\nu}\left(\theta_p\right) = N_p^a = \left(1 - z_p\right)m + z_p X_{pzw}$
This result can be found in [4]. The credibility premiums at contract level are given in the following theorem:
Theorem 1.2 (Credibility estimate at contract level) *Under the same hypotheses as the previous theorem, the following linearized non-homogeneous estimator is obtained for the pure net risk premium of the contract* (p, j):
$\hat{\mu}\left(\theta_p, \theta_{pj}\right) = M_{pj}^a = \left(1 - z_{pj}\right)m_p + z_{pj} X_{pjw}$
This result can be found in [4]. To be able to use the results from this section, one still has to estimate the unknown structure parameters m, m_p, a, b and s^2, appearing in N_p^a and M_{pj}^a. Note that because of the assumptions, we have $m = m_p$. Some unbiased estimators are given in the following section.

2. Parameter estimation

Here and the following (see Section 3 and Section 4) we present the main results leaving the detailed computations to the reader. Combining the statistics of all sectors enables us to derive estimates for the structure parameters on the sector level, and also combining the statistics of the different contracts enables us to derive estimates for the structure parameters on the contract level. So we will provide some useful estimators for the structure parameters: m, m_p, a, b and s^2 in the following theorem:
Theorem 2.1: (Unbiasedness of the estimators) *The random variables*:

$$\hat{m}_p = N_p = X_{pzw},$$
$$\hat{m} = N_0 = X_{zzw},$$
$$\hat{s}^2 = \sum_{p,j,r} w_{pjr}\left(X_{pjr} - X_{pjw}\right)^2 / \sum_{p,j}\left(t_{pj} - 1\right)_+,$$
$$\hat{a} = \sum_{p,j} z_{pj}\left(X_{pjw} - X_{pzw}\right)^2 / \sum_{p}\left(k_p - 1\right)_+,$$
$$\hat{b} = \sum_{p} z_p\left(X_{pzw} - X_{zzw}\right)^2 / (P-1)$$

are unbiased (pseudo-) estimators of the corresponding parameters.

For the proof see [1] from the References. This completes the solution to the hierarchical credibility model in case of non-homogeneous linear credibility estimates. When one considers homogeneous linear credibility approximations-see Section 3-, again one obtains the results with the parameters estimated as in the previous theorem.

3. Jewell model for homogeneous credibility estimators

Theorem 3.1: (Linearized homogeneous estimators in hierarchical model) *Under the hypotheses (J_1)-(J_6) the following linearized homogeneous estimators are obtained for the pure net risk premium of the sector and the pure net risk premium of the contract*:

$$\hat{\nu}(\theta_p) = N_p^a = (1 - z_p)X_{zzw} + z_p X_{pzw}$$
$$\hat{\mu}(\theta_p, \theta_{pj}) = M_{pj}^a = (1 - z_{pj})X_{pzw} + z_{pj} X_{pjw}$$

For the proof see [1] from the References.

4. The description of the hierarchical model with three levels

The results of the two-level hierarchical model of Jewell can be summarized as in the following Diagram 2.

	Individual estimator	Collective estimator	Heterogeneity within	Credibility factor
Level 2 (portfolio)	X_{zzw}		$\hat{b} = V_2$	
Level 1 (sectors)	X_{pzw}	X_{zzw}	$\hat{a} = V_1$	z_p
Level 0 (contracts)	X_{pjw}	X_{pzw}	$\hat{s}^2 = V_0$	z_{pj}

Diagram 2. Jewell's hierarchical model with two levels

Note that in this diagram, the collective estimator at level k is the individual estimator at level $k+1$. The numerator of the estimators $V_k, k = 0,1,2$ is a sum of weighted squared differences of observed values minus individual estimates. The denominator is the number of terms in this summation, minus the number of estimated means. Since the weights appearing in the expressions for the estimator V_k depend on V_k, it must be computed using iteration. Indeed we have:

$$V_0 = \sum_{p,j,r} w_{pjr}\left(X_{pjr} - X_{pjw}\right)^2 / \sum_{p,j}\left(t_{pj} - 1\right)_+;$$
$$z_{pj} = V_1 w_{pj.} / \left(V_0 + V_1 w_{pj.}\right);$$
$$V_1 = \sum_{p,j} z_{pj}\left(X_{pjw} - X_{pzw}\right)^2 / \sum_{p}\left(k_p - 1\right)_+;$$
$$z_p = V_2 z_{p.} / \left(V_1 + V_2 z_{p.}\right);$$
$$V_2 = \sum_{p} z_p\left(X_{pzw} - X_{zzw}\right)^2 / (P-1).$$

It is clear that this process can be generalized to any number of levels. One gets a system of equation in the variables $V_0, V_1, V_2, \ldots$ which have to be solved iteratively. To fix idea we will write down explicitly some of the formulae arising when one level, say company level with index c, is added. We then have to

examine observable variables with four indices X_{cpjq}, where c is the index denoting the company, p denotes the sector, j the contract, and q the year of observation. The following structure variables have to be considered, see Diagram 3.
θ_c : at company level,
θ_{cp} : for sector p in company c;
θ_{cpj} : for contract j in sector p of company c.
Of course the X_{cpjq} variables may denote the average of n_{cpjq} contracts grouped together, as follows:

$$X_{cpjq} = \sum_{i=1}^{n_{cpjq}} X_{cpjq}^{(i)} / n_{cpjq}$$

Company	c=1,2,………
Structure variables:	
company level θ_c	$\rightarrow$
sector level $\rightarrow$	θ_{cp}
contract level $\rightarrow$	θ_{cpj}
Contract	(c, p, j)
Observable variables level 0 with associated weights	X_{cpj1} (w_{cpj1}) $\vdots$ X_{cpjt} (w_{cpjt})

Diagram 3. Hierarchical structure with three levels

One is interested in estimates for the following quantities:

$$\mu_1(\theta_c) = E\lfloor \mu_2(\theta_c, \theta_{cp}) \mid \theta_c \rfloor$$
$$\mu_2(\theta_c, \theta_{cp}) = E\lfloor \mu_3(\theta_c, \theta_{cp}, \theta_{cpj}) \mid \theta_c, \theta_{cp} \rfloor$$
$$\mu_3(\theta_c, \theta_{cp}, \theta_{cpj}) = E\lfloor X_{cpjr} \mid \theta_c, \theta_{cp}, \theta_{cpj} \rfloor$$

Once given a θ on a certain level, one supposes the conditional distribution of the variables appearing on the lower level, to be independent. Then one can aggregate the data with the appropriate weights, starting from the given weights w, and next considering the relevant credibility weights. The results for the multi-level model can be directly derived from the two-step model. Therefore the results are written down immediately. Starting from the weights, one introduces analogous for the covariance matrices as in the case of the two-level model, namely:
$Cov\lfloor \underline{X}_{cpj} \mid \theta_c, \theta_{cp}, \theta_{cpj} \rfloor = \sigma^2(\theta_c, \theta_{cp}, \theta_{cpj}) W_{cpj}$, where $(W_{cpj})_{r,r'} = \delta_{rr'} / w_{cpjr}$. Considering also: $E\lfloor X_{cpjr} \mid \theta_c, \theta_{cp}, \theta_{cpj} \rfloor = \mu(\theta_c, \theta_{cp}, \theta_{cpj})$ one then obtains the credibility factor z_{cpj} for the contract (c, p, j), z_{cp} for the sector (c, p) and z_c for the company c. The summations over the different indices are now calculated by the following conventions:

$$X_{zzzw} = \sum_c \frac{z_c}{z_.} X_{czzw} = \sum_c \frac{z_c}{z_.} \sum_p \frac{z_{cp}}{z_{c.}} X_{cpzw} = \sum_c \frac{z_c}{z_.} \sum_p \frac{z_{cp}}{z_{c.}} \sum_j \frac{z_{cpj}}{z_{cp.}} X_{cpjw} =$$
$$= \sum_c \frac{z_c}{z_.} \sum_p \frac{z_{cp}}{z_{c.}} \sum_j \frac{z_{cpj}}{z_{cp}} \sum_r \frac{w_{cpjr}}{w_{cpj.}} X_{cpjr}$$

Note that the classical estimate (in case no credibility theory is applied) for the expectation m normally is calculated as:

$$\sum_c \sum_p \sum_j \sum_r \frac{w_{cpjr}}{w_{....}} X_{cpjr}$$

The formulae explain why for some portfolios the collective estimator obtained by credibility theory is different from the one obtained by applying averaging procedures. Finally the following scheme to be computed iteratively is obtained. In Diagram 4 the following symbols are used: $V_0 = \sum w_{cpjq}\left(X_{cpjq} - X_{cpjw}\right)^2$ /(Number of terms in this summation minus number of estimated means X_{cpjw});

$$z_{cpj} = V_1 w_{vpj.} / \left(V_0 + V_1 w_{cpj.}\right);$$

$V_1 = \sum_{c,p,j} z_{cpj}\left(X_{cpjw} - X_{cpzw}\right)^2$ / (Number of terms in this summation minus number of estimated means X_{cpzw});

$$z_{cp} = V_2 z_{cp.} / \left(V_1 + V_2 z_{cp.}\right);$$

$V_2 = \sum_{c,p} z_{cp}\left(X_{cpzw} - X_{czzw}\right)^2$ / (Number of terms in this summation minus number of estimated means X_{czzw});

$$z_c = V_3 z_{c.} / \left(V_2 + V_3 z_{c.}\right);$$

$V_3 = \sum_c z_c\left(X_{czzw} - X_{zzzw}\right)^2$ / (Number of terms in this summation minus number of estimated means X_{zzzw})

	Individual estimator	Collective estimator	Heterogeneity within	Credibility factor
Level 3	X_{zzzw}		V_3	
Level 2 (company)	X_{czzw}	X_{zzzw}	V_2	z_c
Level 1 (sectors)	X_{cpzw}	X_{czzw}	V_1	z_{cp}
Level 0 (contracts)	X_{cpjw}	X_{cpzw}	V_0	z_{cpj}

Diagram 4. Jewell's hierarchical model with three levels

So, credibility estimate on the company level is: $\hat{\mu}_1(\theta_c) = (1 - z_c)m + z_c X_{czzw}$;

Credibility estimate on the sector level is:

$$\hat{\mu}_2 = \left(1 - z_{cp}\right)m_c + z_{cp} X_{cpzw};$$

Credibility estimate for the contract level is:

$$\hat{\mu}_3\left(\theta_c, \theta_{cp}, \theta_{cpj}\right) = \left(1 - z_{cpj}\right)m_{cp} + z_{cpj} X_{cpjw}.$$

Linearized homogeneous estimators in the three-level hierarchical model of Jewell are:

$$\hat{\mu}_1(\theta_c) = (1 - z_c)X_{zzzw} + z_c X_{czzw};$$

$$\hat{\mu}_2\left(\theta_c, \theta_{cp}\right) = \left(1 - z_{cp}\right)X_{czzw} + z_{cp} X_{cpzw};$$

$$\hat{\mu}_3\left(\theta_c, \theta_{cp}, \theta_{cpj}\right) = \left(1 - z_{cpj}\right)X_{cpzw} + z_{cpj} X_{cpjw}.$$

V_0 can be calculated directly from the given data. V_1 contains z_{cpj} as weights, therefore it must be calculated iteratively starting with a set of initial values of z_{cpj}, e.g. all equal to 0,5. Using these starting values, a first approximation to V_1 is calculated, which produces a new set of z_{cpj}, and one restarts. In the end, convergence is obtained and one has the value of V_1 and of the z_{cpj}. Next V_2 can be calculated using a similar procedure, and so on.

Conclusions

The credibility method dealt with in this paper is the greatest accuracy theory.

In the first section we demonstrated that the estimators obtained for the pure net risk premium on sector level and for the pure net risk premium on contract level are the best linearized credibility estimators for the two-level hierarchical model of Jewell, using the great-

est accuracy theory.
Section 2 completes the solution of Jewell's hierachical model with two levels in case of non-homogeneous linear credibility estimates. The mathematical theory provides the means to calculate useful estimators for the structure parameters. The property of unbiasedness of these estimators is very appealing and very attractive from point of view practical.
In section 2 we give unbiased estimators for the structural parameters, such that if the structure parameters in the optimal linearized credibility premium are replaced by these estimators, a homogeneous estimator results.
In section 3 we demonstrated that this last estimator is in fact the optimal linearized homogeneous credibility estimator.
In Section 4 we show that the hierarchical structure of Jewell, which is a two level classification procedure, can be generalized to any number of levels. So, in this section we show that one might create a multi-level hierarchical model by, e,g. grouping sectors into cohorts and have different structure parameters for each level. The credibility results for the multi –level model can be directly derived from the two-step model.
So the article provides the mans to calculate the credibility premiums at company level, at sector level and so on, which represents the most recent developments in Bayesian credibility theory. They certainly present the only solution where insurance industry faces risks with risk characteristics that cannot be assigned to any established collective or with a risk coverage under circumstances not earlier met.

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