Semi-linear Credibility Results

Virginia ATANASIU
Academy of Economic Studies, Bucharest

An original paper which suggests a way of thinking for semi-linear credibility theory development, founded on analysis of the functions of the observable random variables.
This line of thought fits perfectly within the framework of the greatest accuracy credibility theory.
Key words: *linear functions, the transformed observations, semi-linear credibility estimators.*

Introduction

Semi-linear credibility estimators are linear functions of transformed observations. The estimators mainly considered here - in the first section - are linear functions of several functions $f_1,...,f_r$ of the observable random variables. So instead of considering linear combinations of the observable variables themselves, one could consider as estimators the class of linear combinations of given functions of the observable variables, and solve - see the second section-:

$$\text{MinE}\left\{\left[f_0(X_{t+1})-\sum_{p=1}^{n}\sum_{r=1}^{t}c_{pr}f_{pr}(X_r)\right]^2\right\}$$

In this way one obtains semi-linear credibility results. One may either assume the functions f_0, f_{pr} to be given in advance, or one may try to determine the best choice. If one wants to estimate a variance-like term, it might be appropriate to consider a quadratic function $f_l = f_{pr}$. So, in case one would like to estimate the variance, a quadratic function is considered. Probably it is also better to take quadratic functions of the observable variables than to approximate by a combination of linear functions. This is also more in harmony with the dimensions of the problem. In some cases one only has data on large claims, so one takes x, if $x > \alpha$, 0 otherwise as the claim amount. One possible choice for f is min(x,d), where d denotes the threshold value above which a claim is called "large". This special choice enables us to evaluate the effect of reinsurance on the risk premium. Sometimes the results of credibility are too sensitive to changes in large claims. Choosing f like this gives us the possibility to avoid too large fluctuations in the premiums. This choice should be considered in combination with an excess of loss reinsurance treaty with retention d for future operations. This article is devoted to semi-linear credibility, where one examines functions of the random variables representing claim amounts, rather than the claim amounts themselves.

Several approximating functions

Here and in the following we present the main results leaving the detailed computations to the reader. Consider a finite sequence θ, X_1,..., X_t, X_{t+1} of random variables. Assume that for fixed θ, the variables X_1,..., X_{t+1} are conditionally independent and identically distributed (i.i.d.). The variables X_1,...,X_t are observable, and θ is the structure variable. The variable X_{t+1} is considered as being not (yet) observable. We assume that $f_p(X_r)$, $p=\overline{0,n}, r=\overline{1,t+1}$ have finite variance. For f_0 we take the function of X_{t+1} we want to forecast. We use the notation:

$$\mu_p(\theta) = E[f_p(X_r)|\theta] \quad (1.1)$$
$$(p=\overline{0,n}, r=\overline{1,t+1})$$

This expression does not depend on r. Our problem is the determination of the linear combination of 1 and the random variables:

$f_p(X_r)$ $(p=\overline{1,n}, r=\overline{1,t})$ (1.2)

closest to $\mu_0(\theta)=E[f_0(X_{t+1})\,|\,\theta]$ in the least squares sense. It is equivalent to determine the linear combination of that form closest to $f_0(X_{t+1})$. For this model we define the following structure parameters:

$m_p=E\lfloor\mu_p(\theta)\rfloor=E\{E\lfloor f_p(X_r)\,|\,\theta\rfloor\}=E\lfloor f_p(X_r)\rfloor$ (1.3)

$a_{pq}=E\{Cov\lfloor f_p(X_r), f_q(X_r)\,|\,\theta\rfloor\}$ (1.4)

$b_{pq} = Cov\left[\mu_p(\theta), \mu_q(\theta)\right]$ (1.5)
$c_{pq} = Cov\left[f_p(X_r), f_q(X_r)\right]$ (1.6)
$d_{pq} = Cov\left[f_p(X_r), \mu_q(\theta)\right]$ (1.7)
for p, q $=\overline{0,n}$. These expressions do not depend on r $=\overline{1,t+1}$. The structure parameters are connected by the following relations:

$$c_{pq} = \mathrm{Cov}\left[f_p(X_r), f_q(X_r)\right] = E\{Cov[f_p(X_r), f_q(X_r) \mid \theta]\} + \mathrm{Cov}\{E[f_p(X_r) \mid \theta],$$
$$, E[f_q(X_r) \mid \theta]\} = a_{pq} + \mathrm{Cov}\left[\mu_p(\theta), \mu_q(\theta)\right] = a_{pq} + b_{pq}\left(p, q = \overline{0,n}\right)$$

So the verification of equality (1.8) is readily performed. Next, we have:

$$d_{pq} = \mathrm{Cov}\left[f_p(X_r), \mu_q(\theta)\right] = E\{Cov[f_p(X_r), \mu_q(\theta) \mid \theta]\} + \mathrm{Cov}\{E[f_p(X_r) \mid \theta],$$
$$E[\mu_q(\theta) \mid \theta]\} = \mathrm{E}\{E[f_p(X_r)\mu_q(\theta) \mid \theta]\} - E[f_p(X_r) \mid \theta]E[\mu_q(\theta) \mid \theta]\} +$$
$$+\mathrm{Cov}\left[\mu_p(\theta), \mu_q(\theta)\right] = E\{\mu_q(\theta)E[f_p(X_r) \mid \theta] - \mu_p(\theta)\mu_q(\theta)\} + b_{pq} = \mathrm{E}\left[\mu_q(\theta)\right.$$
$$\left.\cdot\, \mu_p(\theta) - \mu_p(\theta)\mu_q(\theta)\right] + b_{pq} = E(0) + b_{pq} = 0 + b_{pq} = b_{pq}, (p, q = \overline{0,n})$$

Therefore the verification of equality (1.9) is readily performed.

$c_{pq} = a_{pq} + b_{pq}$ (1.8)
$d_{pq} = b_{pq}$ (1.9)
for p, q $=\overline{0,n}$. This follows from the covariance relations obtained in the probability theory, where they are very well-known. Indeed, we have:

Optimal non-homogeneous linearized estimator

Just as in the case of considering linear combinations of the observable variables themselves, we can also obtain non-homogeneous credibility estimates, taking as estimators the class of linear combinations of given functions of the observable variables, as shown in the following theorem:

Theorem 2.1 (optimal non-homogeneous linearized estimator)
The linear combination of 1 and the random variables $f_p(X_r)$ (p = $\overline{1,n}, r = \overline{1,t}$) closest to $\mu_0(\theta) = E[f_0(X_{t+1}) \mid \theta]$ and to $f_0(X_{t+1})$ in the least squares sense equals:

$$M = \sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p \quad (2.1),$$

where $z_1,...,z_n$ is a solution of the linear system of equations:

$$\sum_{p=1}^{n}\left[c_{pq} + (t-1)d_{pq}\right]z_p = td_{0q} \quad (q = \overline{1,n}) \quad (2.2)$$

or of the equivalent linear system of equations:

$$\sum_{p=1}^{n}\left(a_{pq} + tb_{pq}\right)z_p = tb_{0q} \quad (q = \overline{1,n}) \quad (2.3)$$

Remark 2.1 Inserting the relations (1.8) and (1.9) into (2.2) we observe that (2.2) is equivalent to (2.3). Indeed, let q be fixed. We have:

$$\sum_{p=1}^{n}\left[c_{pq} + (t-1)d_{pq}\right]z_p = td_{0q} \Leftrightarrow \sum_{p=1}^{n}\left[a_{pq} + b_{pq} + (t-1)b_{pq}\right]z_p = tb_{0q} \Leftrightarrow \sum_{p=1}^{n}\left(a_{pq} +\right.$$
$$\left.+b_{pq} + tb_{pq} - b_{pq}\right)z_p = tb_{0q} \Leftrightarrow \sum_{p=1}^{n}\left(a_{pq} + tb_{pq}\right)z_p = tb_{0q}$$

Proof of Theorem 2.1: we have to examine the solution to the problem:

$$\underset{\alpha_0, \alpha}{Min}\, E\left\{\left[\mu_0(\theta) - \alpha_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} f_p(X_r)\alpha_{pr}\right]^2\right\}$$

(2.4),
where: $\alpha = (\alpha_{pr})_{p,r}$. Since (2.4) is the minimum of a positive definite quadratic form, it suffices to find a solution with all partial derivatives equal to zero.
Let:

$$\text{f} = \text{f}(\alpha_0, \alpha_{pq}; p=\overline{1,n}, r=\overline{1,t}) = E\left\{\left[\mu_0(\theta) - \alpha_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} f_p(X_r)\alpha_{pr}\right]^2\right\} =$$

$$= E\left[\mu_0^2(\theta) + \alpha_0^2 + \left(\sum_{p,r} f_p(X_r)\alpha_{pr}\right)^2 - 2\mu_0(\theta)\alpha_0 - 2\mu_0(\theta)\sum_{p,r} f_p(X_r)\alpha_{pr} + \right.$$

$$\left. + 2\alpha_0 \sum_{p,r} f_p(X_r)\alpha_{pr}\right] = \text{E}\left[\mu_0^2(\theta)\right] + \alpha_0^2 + E\left[\left(\sum_{p,r} f_p(X_r)\alpha_{pr}\right)^2\right] - 2\alpha_0 E[\mu_0(\theta)] -$$

$$2\sum_{p,r} E[\mu_0(\theta) f_p(X_r)]\alpha_{pr} + 2\alpha_0 \sum_{p,r} E[f_p(X_r)]\alpha_{pr}$$

Taking the partial derivative with respect to α_0 gives the following equation: $2\alpha_0 - 2E[\mu_0(\theta)] + 2\sum_{p,r} E[f_p(X_r)]\alpha_{pr} = 0$, or:

$$\alpha_0 - E[\mu_0(\theta)] + \sum_{p,r} E[f_p(X_r)]\alpha_{p,r} = 0 \quad (2.5)$$

Using (1.3) in (2.5) we obtain

$$\alpha_0 = m_0 - \sum_{p,r} \alpha_{pr} m_p \quad (2.6)$$

Inserting this expression for α_0 into (2.4) leads to the following problem:

$$\underset{\alpha}{Min}\, E\left(\left\{\mu_0(\theta) - m_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} \alpha_{pr}\left[f_p(X_r) - m_p\right]\right\}^2\right) \quad (2.7)$$

Let: $f(\alpha_{pr}; p=\overline{1,n}, r=\overline{1,t}) = E\left[\left\{\mu_0(\theta) - m_0 - \sum_{p,r} \alpha_{pr}\left[f_p(X_r) - m_p\right]\right\}^2\right] =$

$$= E\left\{\mu_0^2(\theta) + m_0^2 + \sum_{p,r} \alpha_{pr}^2\left[f_p(X_r) - m_p\right]^2 + 2\sum_{p.p'}\sum_{r,r'} \alpha_{pr}\alpha_{p'r'}\left[f_p(X_r) - m_p\right]\cdot\right.$$

$$\cdot\left[f_{p'}(X_{r'}) - m_{p'}\right] - 2m_0\mu_0(\theta) - 2\mu_0(\theta)\sum_{p,r}\alpha_{pr}\left[f_p(X_r) - m_p\right] + 2m_0\sum_{p,r}\alpha_{pr}\cdot$$

$$\left.\cdot\left[f_p(X_r) - m_p\right]\right\} = E\left[\mu_0^2(\theta)\right] + m_0^2 + \sum_{p,r}\alpha_{pr}^2 E\left\{\left[f_p(X_r) - m_p\right]^2\right\} + 2\sum_{p,p'}\sum_{r,r'}\alpha_{pr}\cdot$$

$$\cdot\alpha_{p'r'} E\left\{\left[f_p(X_r) - m_p\right]\left[f_{p'}(X_{r'}) - m_{p'}\right]\right\} - 2m_0 E[\mu_0(\theta)] - 2\sum_{p,r}\alpha_{pr} E[\mu_0(\theta)\cdot$$

$$\cdot\left(f_p(X_r) - m_p\right)] + 2m_0\sum_{p,r}\alpha_{pr} E\left[f_p(X_r) - m_p\right] = E\left[\mu_0^2(\theta)\right] + m_0^2 + \ldots +$$

$$+ \alpha_{pr}^2 E\left\{\left[f_p(X_r) - m_p\right]^2\right\} + \ldots + 2\Big(\ldots + \alpha_{pr}\sum_{\substack{p',r' \\ p'\neq p \\ r'\neq r}}\alpha_{p'r'} E\left\{\left[f_p(X_r) - m_p\right]\left[f_{p'}(X_{r'}) -\right.\right.$$

$$\left.\left.- m_{p'}\right]\right\} + \ldots\Big) - 2m_0 E[\mu_0(\theta)] - 2\left(\ldots + \alpha_{pr} E\left[\mu_0(\theta)\left(f_p(X_r) - m_p\right)\right] + \ldots\right) + 2m_0\cdot$$

$$\cdot\left(\ldots + \alpha_{pr} E\left[\left(f_p(X_r) - m_p\right)\right] + \ldots\right)$$

On putting the derivatives with respect to α_{pr} equal to zero, we get the following system of equations:

$$2\alpha_{pr} E\left\{\left[f_p(X_r) - m_p\right]^2\right\} + 2\sum_{\substack{p',r' \\ p'\neq p \\ r'\neq r}}\alpha_{p'r'} E\left\{\left[f_p(X_r) - m_p\right]\left[f_{p'}(X_{r'}) - m_{p'}\right]\right\} - 2$$

$$\cdot E\left[\mu_0(\theta)\left(f_p(X_r) - m_p\right)\right] + 2m_0 E\left[f_p(X_r) - m_p\right] = 0, \left(p=\overline{1,n}; r=\overline{1,t}\right), \text{ that is:}$$

$E\{[\mu_0(\theta)-m_0][f_p(X_r)-m_p]\}=\sum_{p',r'}\alpha_{p'r'}E\{[f_p(X_r)-m_p][f_{p'}(X_{r'})-m_{p'}]\}$ (p = $\overline{1,n}, r=\overline{1,t}$), or:

$E\{[\mu_0(\theta)-E(\mu_0(\theta))][f_p(X_r)-E(f_p(X_r))]\}=\sum_{p',r'}\alpha_{p'r'}E\{[f_p(X_r)-\mathrm{E}(f_p(X_r))]$

$\cdot[f_{p'}(X_{r'})-E(f_{p'}(X_{r'}))]\},(p=\overline{1,n},r=\overline{1,t})$

So: $Cov\ [\mu_0(\theta),f_p(X_r)]=\sum_{p',r'}\alpha_{p'r'}Cov[f_p(X_r),f_{p'}(X_{r'})]$ (2.8)

$(p=\overline{1,n},r=\overline{1,t})$. Substituting p with q, r with r' in (2.8) one obtains:

$Cov\ [\mu_0(\theta),f_q(X_{r'})]=\sum_{p'=1}^{n}\sum_{r'=1}^{t}\alpha_{p'r'}Cov[f_q(X_{r'}),f_{p'}(X_{r'})]$ (2.9)

$(q=\overline{1,n},r'=\overline{1,t})$. Substituting p' with p,r' with r in (2.9) one obtains:

$Cov[\mu_0(\theta),f_q(X_{r'})]=\sum_{p=1}^{n}\sum_{r=1}^{t}\alpha_{pr}Cov[f_p(X_r),f_q(X_{r'})],(q=\overline{1,n},r'=\overline{1,t})$ (2.10)

From (1.5), (1.7), (1.9) we get

$Cov\ [\mu_0(\theta),f_q(X_{r'})]=Cov[f_q(X_{r'}),\mu_0(\theta)]=d_{q0}=b_{q0}=Cov[\mu_q(\theta),\mu_0(\theta)]=$

$=Cov\ [\mu_0(\theta),\mu_q(\theta)]=b_{0q}(=d_{0q})$ (2.11),

where q $=\overline{1,n},r'=\overline{1,t}$. Let r, r' $=\overline{1,t}$ with r $\neq r'$. From (1.1), (1.5), (1.9) we get

$Cov\ [f_p(X_r),f_q(X_{r'})]=E[f_p(X_r)f_q(X_{r'})]-E[f_p(X_r)]E[f_q(X_{r'})]=$

$=\mathrm{E}\{E[f_p(X_r)f_q(X_{r'})\mid\theta]\}-E\{E[f_p(X_r)\mid\theta]\}E\{E[f_q(X_{r'})\mid\theta]\}=$

$=\mathrm{E}\{E[f_p(X_r)\mid\theta]E[f_q(X_{r'})\mid\theta]\}-E[\mu_p(\theta)]E[\mu_q(\theta)]=E[\mu_p(\theta)\mu_q(\theta)]-$

$-\mathrm{E}[\mu_p(\theta)]E[\mu_q(\theta)]=Cov[\mu_p(\theta),\mu_q(\theta)]=b_{pq}=d_{pq}$ (2.12),

for all p, q = $\overline{1,n}$.

In view of (1.1), (1.4), (1.5), (1.8) and (2.12), we calculate the summation:

$\sum_{r=1}^{t}\alpha_{pr}Cov[f_p(X_r),f_q(X_{r'})]=\alpha_{pr'}Cov[f_p(X_{r'}),f_q(X_{r'})]+\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}\cdot\mathrm{Cov}[f_p(X_r),$

$,f_q(X_{r'})]=\alpha_{pr'}\{E[Cov(f_p(X_{r'}),f_q(X_{r'})\mid\theta)]+Cov[E(f_p(X_{r'})\mid\theta),\mathrm{E}(f_q(X_{r'})\mid$

$\mid\theta)]\}+\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}d_{pq}=\alpha_{pr'}\{a_{pq}+Cov[\mu_p(\theta),\mu_q(\theta)]\}+d_{pq}\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}=\alpha_{pr'}(a_{pq}+b_{pq})+$

$+d_{pq}\cdot\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}=\alpha_{pr'}c_{pq}+d_{pq}\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}$ (2.13),

where: p,q = $\overline{1,n},r'=\overline{1,t}$.

Introducing (2.11) and (2.10) in (2.13), one obtains:

$$b_{0q}=\sum_{p=1}^{n}\left[\alpha_{pr'}c_{pq}+d_{pq}\sum_{\substack{r\\ r\neq r'}}\alpha_{pr}\right],(q=\overline{1,n},r'=\overline{1,t})\quad(2.14)$$

Because of the symmetry in time clearly:

$\alpha_{p1}=\alpha_{p2}=...=\alpha_{pt}=\alpha_p$ (2.15)

and so the system of equations (2.14) can be written as:

$b_{0q} = \sum_{p=1}^{n} \alpha_p \left[c_{pq} + (t-1)d_{pq}\right], (q = \overline{1,n})$, or (see (1.9)):

$d_{0q} = \sum_{p=1}^{n} \alpha_p \left[c_{pq} + (t-1)d_{pq}\right], (q = \overline{1,n})$ (2.16)

Comparing (2.2) with (2.16), we conclude that:

$\alpha_p = z_p / t$ (p = $\overline{1,n}$) (2.17)

Remark 2.2 The relation (2.15) can be proven as follows; for r' = 1, r' = 2,…., r' = t-1, r' = t the system of equations (2.14) implies:

$b_{0q} = \sum_{p=1}^{n} \left(\alpha_{p1} c_{pq} + \alpha_{p2} d_{pq} + ... + \alpha_{pt} d_{pq}\right); q = \overline{1,n},$

$b_{0q} = \sum_{p=1}^{n} \left(\alpha_{p2} c_{pq} + \alpha_{p1} d_{pq} + \alpha_{p3} d_{pq} + ... + \alpha_{pt} d_{pq}\right); q = \overline{1,n},$

$b_{0q} = \sum_{p=1}^{n} \left(\alpha_{p,t-1} c_{pq} + \alpha_{p1} d_{pq} + ... + \alpha_{p,t-2} d_{pq} + \alpha_{pt} d_{pq}\right); q = \overline{1,n},$

$b_{0q} = \sum_{p=1}^{n} \left(\alpha_{pt} c_{pq} + \alpha_{p1} d_{pq} + \alpha_{p2} d_{pq} + ... + \alpha_{p,t-1} d_{pq}\right); q = \overline{1,n}$.

The first two equalities imply:

$$\sum_p \left(\alpha_{p1} c_{pq} + \alpha_{p2} d_{pq} + ... + \alpha_{pt} d_{pq}\right) = \sum_p \left(\alpha_{p1} d_{pq} + \alpha_{p2} c_{pq} + \alpha_{p3} d_{pq} + \ldots + \right.$$
$$\left. + \alpha_{pt} d_{pq}\right)$$, that is:

$$\sum_p \alpha_{p1} c_{pq} + \sum_p \alpha_{p2} d_{pq} + \sum_p \left(\alpha_{p3} d_{pq} + ... + \alpha_{pt} d_{pq}\right) = \sum_p \alpha_{p1} d_{pq} + \sum_p \alpha_{p2} c_{pq} +$$
$$+ \sum_p \left(\alpha_{p3} d_{pq} + ... + \alpha_{pt} d_{pq}\right), \forall q = \overline{1,n}$$, or:

$$\sum_{p=1}^{n} \alpha_{p1} \left(c_{pq} - d_{pq}\right) = \sum_{p=1}^{n} \alpha_{p2} \left(c_{pq} - d_{pq}\right), \forall q = \overline{1,n},$$

that is (see (1.8) and (1.9)):

$$\sum_{p=1}^{n} \alpha_{p1} a_{pq} = \sum_{p=1}^{n} \alpha_{p2} a_{pq}, \forall q = \overline{1,n} \quad (2.18)$$

Hence, for q = 1 we have: $\sum_{p=1}^{n} \alpha_{p1} a_{p1} = \sum_{p=1}^{n} \alpha_{p2} a_{p1}, \forall n \in \mathbb{N}^*$ and from here for n=1, we get: $\alpha_{11} a_{11} = \alpha_{12} a_{11}$, so:

$\alpha_{11} = \alpha_{12}$ (2.19),

because $a_{11} \neq 0$.

Next, for q = 2 one obtains: $\sum_{p=1}^{n} \alpha_{p1} a_{p2} = \sum_{p=1}^{n} \alpha_{p2} a_{p2}, \forall n \in \mathbb{N}^*$ and from here for n = 2, we get: $\alpha_{11} a_{12} + \alpha_{21} a_{22} = \alpha_{12} a_{12} + \alpha_{22} a_{22}$ that is (see (2.19)): $\alpha_{21} a_{22} = \alpha_{22} a_{22}$, so:

$\alpha_{21} = \alpha_{22}$ (2.20),

because $a_{22} \neq 0$.

Next, for q = 3 one obtains: $\sum_{p=1}^{n} \alpha_{p1} a_{p3} = \sum_{p=1}^{n} \alpha_{p2} a_{p3}, \forall n \in \mathbb{N}^*$ and from here for n = 3, we get: $\alpha_{11} a_{13} + \alpha_{21} a_{23} + \alpha_{31} a_{33} = \alpha_{12} a_{13} + \alpha_{22} a_{23} + \alpha_{32} a_{33}$, that is (see (2.19), (2.20)): $\alpha_{31} a_{33} = \alpha_{32} a_{33}$, so:

$\alpha_{31} = \alpha_{32}$ (2.21),

because $a_{33} \neq 0$.

From (2.19), (2.20) and (2.21), we may conclude that:

$\alpha_{p1} = \alpha_{p2}$, for all p = $\overline{1,n}$ (2.22)

The equality (2.22) can be proven by induction. Indeed, for n = 1 is readily performed.

We assume that:

$\alpha_{p1} = \alpha_{p2}, \forall p = \overline{1, n-1}$

From (2.18) for q = n, we have: $\sum_{p=1}^{n} \alpha_{p1} a_{pn} = \sum_{p=1}^{n} \alpha_{p2} a_{pn}$, that is:

$$\alpha_{11}a_{1n} + \alpha_{21}a_{2n} + ... + \alpha_{n-1,1}a_{n-1,n} + \alpha_{n1}a_{nn} = \alpha_{12}a_{1n} + \alpha_{22}a_{2n} + ... + \alpha_{n-1,2}a_{n-1,n} +$$

$+\alpha_{n2}a_{nn}$ and so: $\alpha_{n1}a_{nn} = \alpha_{n2}a_{nn}$. From here, we get: $\alpha_{n1} = \alpha_{n2}$, because $a_{nn} \neq 0$. From the following two equalities and proceeding in the same way, one obtains:

$\alpha_{p2} = \alpha_{p3}$, for all p = $\overline{1, n}$ (2.23)

Finally, from the last two equalities and proceeding in the same way, one obtains:

$\alpha_{p,t-1} = \alpha_{pt}$, for all p = $\overline{1, n}$ (2.24)

The relations (2.22), (2.23), and (2.24) imply (2.15). Now (2.6) and (2.17) lead to:

$$\alpha_0 = m_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} \frac{z_p}{t} m_p = m_0 - \sum_{p=1}^{n} t \frac{z_p}{t} m_p = m_0 - \sum_{p=1}^{n} z_p m_p$$

(2.25)

Consequently:

$$\hat{\mu}_0(\theta) = M = \alpha_0 + \sum_{p=1}^{n}\sum_{r=1}^{t} f_p(X_r)\alpha_{pr} = \sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p$$

,

as was to be proven (see (2.4), (2.15), (2.17) and (2.25)).

For the special case when n=1, **Theorem 2.1** reads:

Theorem 2.2 (optimal non-homogeneous linearized estimator, n = 1)

The linear combination of 1 and the random variables $f_1(X_r)$ (r = $\overline{1, t}$) closest to $\mu_0(\theta)$ and to $f_0(X_{t+1})$ in the least squares sense equals:

$M = z\sum_{r=1}^{t} \frac{1}{t} f_1(X_r) + m_0 - zm_1$ (2.26),

where: $m_1 = E[f_1(X_r)]$, $z = td_{01}/\{c_{11}+(t-1)d_{11}\}$ with: $d_{01} = Cov[f_0(X_r), f_1(X_{r'})]$, $d_{11} = Cov[f_1(X_r), f_1(X_{r'})]$ for $r \neq r'$, $c_{11} = Cov[f_1(X_r), f_1(X_r)]$.

Proof:

For n=1 the relation (1.3) implies: $m_1 = E[f_1(X_r)]$. For n=1 the relation (1.6) implies: $c_{11} = Cov[f_1(X_r), f_1(X_r)]$. For n=1 the linear system of equations (2.2) reads: $[c_{11}+(t-1)d_{11}]z = td_{01}$ which is equivalent to the following equation: $z = td_{01}/\{c_{11}+(t-1)d_{11}\}$. From (2.12) for n=1 one obtains: $d_{01} = Cov[f_0(X_r), f_1(X_{r'})]$, $d_{11} = Cov[f_1(X_r), f_1(X_{r'})]$, where $r \neq r'$. Finally, for n=1 the relation (2.1) reads:

$$M = z\sum_{r=1}^{t} \frac{1}{t} f_1(X_r) + m_0 - zm_1.$$

So the theorem is proven.

Conclusions

This paper is an original approach of a more general credibility model.

We obtained a semi-linear credibility model, which involves the class of linear combinations of given functions of the observable variables, for solving the minimization problems of the type: $MinE\left\{\left[f_0(X_{t+1}) - \sum_{p=1}^{n}\sum_{r=1}^{t} c_{pr} f_{pr}(X_r)\right]^2\right\}$. So the approximation to $f_0(X_{t+1})$ or to $\mu_0(\theta) = E[f_0(X_{t+1}) \mid \theta]$ furnished in this article, is based on prescribed approximating functions $f_1, f_2, \ldots, f_n$. The usefulness of this approximation is that it is easy to apply, since it is sufficient to know estimates for the parameters a_{pq}, b_{pq} appearing in the credibility factors z_p. The estimators mainly considered here are linear functions of several functions $f_1, f_2, \ldots, f_r$ of the observable random variables, which represents claim amounts, rather than the claim amounts themselves. For this reason, semi-linear credibility estimators, which are linear functions of transformed observations lead to easily computable premiums. This semi-linear credibility results are the most recent developments in credibility theory and they certainly present the only solution where insurance industry faces risks with basic risk characteristics that cannot be assigned to any established collective or with a risk coverage under circumstances not earlier met. We give a rather explicit description of the input data for the model used, only to show that in practical situations there will always be enough data to apply semi-linear credibility theory to a real insurance portfolio. The point we want to emphasize is that practical application of semi-linear credibility is feasible nowadays using the greatest accuracy credibility theory.

References

[1]. **Atanasiu V.,** ***Contributions to the credibility theory***, doctoral dissertation, University of Bucharest-Faculty of Mathematics, 2000.

[2]. **Atanasiu V.,** ***Un model de credibilitate,*** Revista Studii şi Cercetări de Calcul Economic şi Cibernetică Economică, XXXII, nr.3, 1998.

[3]. **Goovaerts M.J., Kaas R., Van Heerwaarden, Bauwelinckx T.,** ***Insurance Series,*** volume 3, ***Effective Actuarial Methods,*** University of Amsterdam, The Netherlands, 1991.

[4]. **Pentikäinen T., Daykin C.D.,Pesonen M.,** ***Practical Risk Theory for Actuaries,*** Université Pierré et Marie Curie, 1990.

[5]. **Sundt B.,** ***An Introduction to Non-Life Insurance Mathematics,*** Veröffentlichungen des Instituts für Versicherungswissenschaft der Universität Mannheim Band 28, VVW Karlsruhe, 1990.